Topology of Closed 1-Forms on Manifolds with Boundary

LI, TIEQIANG,TAN

Topology of Closed 1-Forms on Manifolds with Boundary

Tieqiang Li

A Thesis presented for the degree of
Doctor of Philosophy

Topology
Department of Mathematical Sciences
University of Durham
England

November 2009

Topology of Closed 1-Forms on Manifolds with Boundary

Tieqiang Li

Submitted for the degree of Doctor of Philosophy
2009

Abstract

The topological structure of a manifold can be effectively revealed by studying the critical points of a nice function assigned on it. This is the essential motivation of Morse theory and many of its generalisations from a modern viewpoint. One fruitful direction of the generalisation of the theory is to look at the zeros of a closed 1-form which can be viewed locally as a real function up to an additive constant, initiated by S.P. Novikov, see [32] and [33]. Extensive literatures have been devoted to the study of so-called Novikov theory on closed manifolds, which consists of interesting objects such as Novikov complex, Morse-Novikov inequalities and Novikov ring.

On the other hand, the topology of a space, e.g. a manifold, provides vital information on the number of the critical points of a function. Along this line, a whole different approach was suggested in the 1930s by Lusternik and Schnirelman [25] and [26]. M. Farber in [9], [10], [11] and [12] generalised this concept with respect to a closed 1-form, and used it to study the critical points and existence of homoclinic cycles on a closed manifold in much more degenerate settings.

This thesis combines the two aspects in the context of closed 1-forms and attempts a systematic treatment on smooth compact manifolds with boundary in the sense that the transversality assumptions on the boundary is consistent thoroughly.

Overall, the thesis employs a geometric approach to the generalisation of the existing results.

Declaration

The work in this thesis is based on research carried out at the TOPOLOGY Group, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

Acknowledgements

I want to first express my wholehearted gratitude to my family. In particular, thanks to my father who encourages and supports me unreservedly to pursue my academic studies to PhD level; thanks to my mother who inspired me to study in England; thanks to my grandmother who is in heaven now, who brought me up with all her kindness and love.

I am also severely indebted to my supervisors Dr. Dirk Schütz and Prof. Michael Farber. I would like to thank Dirk for his patient instructions and dedications to research and Mathematics, from which I have benefited enormously. I want to thank Prof. Farber for bringing me into the Mathematics research in Algebraic Topology and the conferences I was invited to.

I want to also thank all the other people I have come aross in the world of Mathematics. In particular, my fellow colleagues, Ruth Jenni, Mark Grant, Dingyu Yang, Kathryn Barnett and Joey Oliver (just to name a few). Thanks to Ruth who has such a lively personality and undivided love to Mathematics, even we are seperated into different cults in the wide wild Maths world (she is on Algebraic/Arithmetic Geometry), Ruth always magaged to find common language and laughters to cheer up in those depressive moments one's PhD career sooner or later will run into. Thanks to Mark as my senior and friend for his maturity and seriousness to Maths, especially, the fruitful conversation we had with Prof. Mihai Damian at the dinner table at Oberwolfach, which leads to the exact sequence results in Chapter 4 of this thesis. Thanks to Dingyu, who I am so proud of knowing, and have no doubt his Mathematics one day will shine. And thank-you to Kathryn and Joey for sharing the same office and all those trivial conversations that made my PhD life easier to cope with. Thanks to Pam for all the morning greetings and teas.

Thanks to all my King's Church fellows for all the prayers and meal/tea sharings over the years.

My PhD research project is sponsored by Dorothy Hodgkin Postgraduate Award via Engineering and Physical Sciences Research Council in the UK.

November 24, 2009

Introduction

Here we introduce the essential settings for the whole thesis. In particular, we recall the Morse nondegenerate condition that is central to our studies. Then we outline the content of each chapter and potential contribution to the field.

The manifold we use in this thesis is always smooth, meaning it is equipped with a differentiable structure of class C^∞. Let $f : M \to \mathbb{R}$ be a real function on M, the critical points of f are the points in M where the derivative $\mathrm{d}f$ is singular, i.e. in local coordinates $(x_1, \ldots, x_n)$, $p \in M$ is *critical* if

$$\mathrm{d}f(p) = \frac{\partial f}{\partial x_1}(p)\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}(p)\mathrm{d}x_n = 0,$$

and that is

$$\frac{\partial f}{\partial x_i}(p) = 0 \text{ for all } i = 1, \ldots, n.$$

we denote the set of critical points of f as $\mathrm{Crit}\, f$.

At each critical point p of f, we can define a symmetric bilinear form $f_{**} : T_pM \times T_pM \to \mathbb{R}$ over the tangent vector space as

$$f_{**}(v, w) = V_p(W(f)),$$

where $v, w \in T_pM$ are tangent vectors, and V, W are trivial extensions of v, w as vector fields on M respectively. We call f_{**} the *Hessian* of f at p. In local coordinates with $x = (x_1, \ldots, x_n) \in U_p$ where U_p is a neighbourhood of p, we can write $V = \sum_i a_i \frac{\partial}{\partial x_i}$ and $W = \sum_j b_j \frac{\partial}{\partial x_j}$ where a_i, b_j are constant functions, so that the Hessian f_{**} is written in the following matrix form:

$$\left(a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right).$$

Then we call a critical point p of f *Morse nondegenerate*, if the Hessian f_{**} at p is nondegenerate, meaning that its matrix form has full rank n. Now the Morse

Lemma, in [28, Chapter 2] for instance, tells us that with such nondegeneracy condition, there exists a suitable local coordinate system $(y_1, \ldots, y_n)$ in a neighbourhood U of p with $y_i(p) = 0$ for all i, and

$$f(q) \;=\; f(p) - y_1^2(q) - \cdots - y_\lambda^2(q) + y_{\lambda+1}^2(q) + \cdots + y_n^2(q),$$

for all $q \in U$, where λ is a constant.

Then f_{**} can be diagonalised to the following form:

$$\begin{pmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{pmatrix}.$$

Moreover, the number λ of negative eigenvalues is independent of the choice of local coordinate systems. Again, we refer the details to [28, Chapter 2]. Therefore, for each critical point p, we associate it with an *index* $\mathrm{ind}\,(p)$ as the number of negative eigenvalues of the Hessian f_{**} at p. If all the critical points of f are Morse nondegenerate, we call f a *Morse function.*

Now we state the foundational result of Morse theory that relates the homotopy type of a closed manifold M to the critical points of an associated Morse function f in the following theorem of Morse:

Theorem 0.0.1 If M is a closed manifold and f is a Morse function on M, then M has the homotopy type of a CW-complex, with each cell of dimension i corresponding to a critical point of f with index i.

The theorem effectively summarises the beauty and simplicity of the original Morse theory, and together with the following *Smale transversality* condition, we can read off the cellular structural information of a manifold by simply looking at the critical points and the gradient flow of a Morse function on the manifold.

Given a Riemannian metric $\langle \cdot, \cdot \rangle$ on M, we have a gradient vector field v of f satisfying $\mathrm{d}f(u) = \langle v, u \rangle$ for any vector field $u \in TM$ of M. v generates a global flow on M, let $x \cdot t$ be a trajectory of x following the flow, we introduce the following two objects from dynamics:

$$W^s(p,v) = \{x \in M : x \cdot t \to p \text{ as } t \to +\infty\}$$

is called the *stable manifold* of p and similarly

$$W^u(p,v) = \{x \in M : x \cdot t \to p \text{ as } t \to -\infty\}$$

is called the *unstable manifold* of p. A function f is said to be *transverse* or satisfy the *Smale transversality condition* if any stable manifold $W^s(p,v)$ and unstable manifold $W^u(q,v)$ always intersect transversely (or empty), with $p, q \in \operatorname{Crit} f$. We elaborate this property in the actual studies.

In the Morse situation, topics on the category of manifolds with boundary has been explored with a strong analytic flavour. [4], [43] and [1] have all produced analogous results in Morse inequalities and relative chain complexes via an analytic approach of Witten. In particular [4] has obtained such results in very degenerate conditions for a closed 1-form. In the category situation, [6] and [30] study the relative version of the classical category.

The thesis is implicitly divided into two parts, part one, made up by the first five chapters, is devoted to the development of a relative Morse-Novikov theory with respect to a Morse function or a Morse closed 1-form, and beyond, namely, the Novikov-Bott situation; whereas the sixth chapter alone as part two describes the Lusternik-Schnirelman category approach to the issue of critical points in a relative setting. We give a more detailed breakdown in the following:

Chapter 1 sets up the boundary assumptions of a Morse function f for a manifold M with boundary ∂M, and verify these assumptions by identifying the correct homotopy type of M. It goes further to the construction of the Morse chain complex which lays the foundation for analoguous works in the closed 1-form case.

Chapter 2 continues on the Morse chain complex in a circle-valued function which is equivalent to a rational closed 1-form, where we employ the inverse limit technique of [38]. In Chapter 3, after a rational approximation of a general closed 1-form, we are able to regenerate the construction of boundary maps in Chapter 2, together with the Latour trick, we obtain a relative Novikov complex and its chain homotopy equivalence with the simplicial chain complex.

On the one hand, despite the introductory role of the first three chapters to a relative Morse/Novikov chain complex, the main challenge there is the geometric exposition of how to make the boundary condition compactible with the classical Morse deformation retraction and the chain homotopy. Meanwhile, it is also vital for the exact sequences relating the absolute Morse/Novikov complexes with the relative one, which we detail in Chapter 4.

In Chapter 4, we construct a Morse function/closed 1-form that satisfies the boundary assumptions and derive long exact sequences relating the relative homology with the absolute ones. This leads us further to qan improved Morse-Novikov inequalities.

Then in Chapter 5, we attempt to generalise the nondegeneracy condition in the sense of Bott by constructing a spectral sequence, and pick up the other story line in the development of Morse inequalities. We introduce the no homoclinic cycle condition there in order to complete the geometric picture.

In Chapter 6, after departing from the conventional Morse nondegeneracy condition in Chapter 5, we switch to the Lusternik-Schnirelman category theory with respect to a closed 1-form, which allows rather degenerate behaviour of the critical set. We extend Farber's work to the relative case and show the link to the absolute ones by an inequality.

Chapter 7 concludes the whole thesis.

Contents

Abstract ii

Declaration iii

Acknowledgements iv

Introduction vi

1 A real Morse function on a manifold with boundary 1

1.1 Set up . . . 1

1.2 Homotopy type . . . 5

1.3 Chain complex . . . 10

1.4 Homology and Morse inequalities . . . 20

2 Circle-valued functions 22

2.1 The minimal covering space . . . 22

2.2 The chain complex . . . 23

2.3 The chain homotopy equivalence . . . 27

2.4 Universal covering . . . 28

3 Closed 1-forms 31

3.1 Connection with the Morse complex . . . 31

3.2 Novikov ring . . . 32

3.3 Novikov complex . . . 33

3.4 Continuation . . . 39

3.5 Homotopy equivalence(with the Latour trick) . . . 44

3.6 Morse inequalities 48

4 The exact sequences 50

4.1 On a real Morse function 50

4.1.1 Short exact sequence for $(M, \partial M)$ 51

4.1.2 Another short exact sequence 54

4.1.3 Homology and improved Morse inequalities 56

4.2 On a closed 1-form 58

5 Morse-Bott nondegeneracy 63

5.1 The settings 63

5.2 Spectral sequences and the chain complexes 65

5.2.1 For real functions 65

5.2.2 For circle-valued functions 72

5.2.3 Examples of Novikov-Bott closed 1-forms with the No Homoclinic Cycle Condition 82

6 Lusternik-Schnirelman category with respect to a closed 1-form 84

6.1 Definitions 84

6.2 Properties 91

6.2.1 Inequalities 92

6.2.2 Cup length lower bound 98

6.3 Homoclinic cycles and critical points 103

6.3.1 Main result 106

6.3.2 Proof of null homotopy of U_i in Theorem 6.3.9 112

7 Conclusion 116

Bibliography 117

List of Figures

1.1 Gradient vector field and flow at the cubical neighbourhood of Γ . . . 7

1.2 Deformation with corners 8

1.3 Self-indexing rearrangement 12

1.4 Pushing up the entry set T 13

2.1 $C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}) = \bigoplus_{k=1}^{j} \mathbb{Z}(t^k p)$ 24

2.2 $C_*^{\mathrm{MS}}(M_{j+1}^1, \bar{B}_{j+1} \cup M_{j+1}, \bar{f}, \bar{v}) = \bigoplus_{k=1}^{j+1} \mathbb{Z}(t^k p)$ 24

3.1 The lift of loop γ in the covering 37

3.2 Manifold with corners (the shaded area) 41

3.3 Overlaps of the open cover 46

3.4 Operation near a critical point 47

4.1 Torus with boundary 58

5.1 Choice of $v^{\mu+1}$ and $\Psi_{\mu+1}$ 80

6.1 Homotopy Invariance 86

6.2 Counter-example of (N, C)-movability 88

Chapter 1

A real Morse function on a manifold with boundary

In this first chapter, we set up the assumptions on the boundary of our manifold for the whole thesis, and test them on a real function to get the homotopy equivalence statement and subsequent Morse inequalities, using standard techniques. Moreover, since a closed 1-form is viewed locally as a real function, the validity of these assumptions lays the foundation for the rest of the development of the relative Morse-Novikov theory.

1.1 Set up

Our chief interest here is in the category of smooth compact manifolds with boundary, of which closed manifolds are just special cases when the boundary is empty.

Let M be such a manifold of dimension m, and we denote its boundary as ∂M. Choose a smooth function $f : M \to \mathbb{R}$ with Morse nondegenerate critical points, i.e. a Morse function. Given a Riemannian metric $\langle \cdot, \cdot \rangle$ on M, we equip M with a gradient vector field v of f as $\mathrm{d}f_x(u) = \langle v_x, u_x \rangle$ for each $x \in M$ and vector field $u : M \to TM$. The classical Morse theory empowers one to describe the cellular structure of a closed manifold M by simply studying the critical points of a Morse function on M and the trajectories of its negative gradient vector field. Here, however, the flow generated by v is not necessarily defined globally on $M \times \mathbb{R}$, so the next thing we want to do is to recover a global flow.

To achieve this, we firstly extend M to a larger manifold. Consider the boundary ∂M, and we have the product $\partial M \times [-1,1]$ which inherits the product differentiable structure of ∂M and $[-1,1] \subset \mathbb{R}$, then choose an interior collaring for ∂M in M, and identify it with the positive half $\partial M \times [0,1]$ of $\partial M \times [-1,1]$. In particular, we have $\partial M \cong \partial M \times \{0\}$. Now define $M^+ = M \cup_{\partial M} \partial M \times [-1,0]$, and f can also be extended smoothly to M^+. With a slight abuse of notation, we use the same f for this function on M^+. Fixing a Riemannian metric on M^+, we have a gradient vector field on M^+. Furthermore, this gradient can be made to vanish near $\partial M \times \{-1\}$ in a smooth fashion, denoted v^+. Having done all these, the negative flow generated by $-v^+$ is complete on $M^+ \times \mathbb{R}$. To be precise, we denote $\Phi : M^+ \times \mathbb{R} \to M^+$ as the negative flow generated by $-v^+$ and often use the shorthand notation $x \cdot t = \Phi(x,t)$.

Remark 1.1.1 With the latter Riemannian metric restricted to M, $v^+|_M$ will also be a gradient vector field of $f|_M$ restricted to M.

Suppose we have the product differentiable structure on a tubular neighbourhood $\partial M \times [-1,1]$ of the boundary ∂M in M^+, then any two local coordinate systems will have the nth coordinate independent of the rest. Let $(U, x_1, \ldots, x_{n-1}, t)$ and $(V, y_1, \ldots, y_{n-1}, s)$ be two local coordinate systems centred at $z \in \partial M$, then

$$-\frac{\partial f}{\partial t}(z) \leq 0 \Longleftrightarrow -\frac{\partial f}{\partial s}(z) \leq 0.$$

And s can be seen as a multiple of t by some positive function of x: $s = K(x)t$ because for any two coordinate systems, they have to respect the embeddings $\partial M \times [-1,1] \hookrightarrow M^+$ with $\partial M \times [0,1] \subset M$, where the orientation of the last coordinate $[-1,1]$ is fixed. Bearing this in mind, we have the following assumptions on f.

Assumptions on f

After choosing a differentiable structure on M^+, we assume the following:

A1 The function f has no critical point on ∂M. This implies that f has no critical points in a neighbourhood of ∂M. Without loss of generality we assume that f has no critical points in the interior of the entire collaring $\partial M \times [-1,1]$.

A2 By the construction of M^+, we have a readily chosen tubular collaring $\partial M \times [-1, 1]$ of ∂M. We assume the partial derivative $\frac{\partial f}{\partial t}$, where t is the coordinate for $[-1, 1]$, is a smooth function on $\partial M \times \{0\}$ and zero is a regular value of $\frac{\partial f}{\partial t}(x, 0)$. Denote $\Gamma = \{x \in \partial M : \frac{\partial f}{\partial t}(x, 0) = 0\}$, this is equivalent to say Γ is a 1-codimensional closed submanifold of ∂M.

Definition 1.1.2 We define the *exit set* B *of* f as:

$$B = \{x \in \partial M : -\frac{\partial f}{\partial t}(x, 0) \leq 0\} \subset \partial M.$$

A3 Fix a tubular collaring of Γ in ∂M, $\Gamma \times [-1, 1] \subset \partial M$, so that $\Gamma \times [-1, 0] \subset B$ with $\Gamma \times \{0\} \cong \Gamma$. Then when a point lies in the cubical neighbourhood of Γ in M^+, we write it in local coordinates:

$$(x, s, t) \in \Gamma \times [-1, 1] \times [-1, 1],$$

where $x = (x_1, \cdots, x_{m-2})$, then we assume

$$\frac{\partial f}{\partial s}(x, 0, 0) > 0.$$

Notice that the conditions $\mathbf{A1}, \mathbf{A2}$ and $\mathbf{A3}$ do not depend on the particular choice of collarings. However, in order to get that B serves as the exit set for the negative gradient flow, we need a restriction on the gradients. We formalise the idea by the following notion:

Definition 1.1.3 Let f be a Morse function that satisfies $\mathbf{A1}, \mathbf{A2}$ and $\mathbf{A3}$. A gradient v of f is called *transverse on* ∂M if the Riemannian metric is a product metric on Γ and on ∂M with respect to the same tubular neighbourhoods as in **A2** and **A3**.

The above definition can be easily adapted to Morse 1-form in the subsequent chapters.

This will guarantee the negative flow of such gradient exits from the interior of B only. We elaborate this property in Section 1.2.

Remark 1.1.4 We will keep this particular choice of gradient vector field on manifold with boundary consistent throughout the thesis. So all the gradient vector fields on manifolds with boundary is automatically chosen to be transverse in the rest of the thesis, unless it is otherwise specified.

Remark 1.1.5 The enlargement of M to M^+ is technically an auxiliary consideration, to ensure the negative flow is defined over the whole original manifold M including its boundary and so improves the description of the "flowing out" nature of B.

Remark 1.1.6 The functions satisfy assumptions **A1** and **A2** are generic, we can always obtain such ones by small perturbation near the boundary. Whereas **A3** has a more restrictive nature, in return, it guarantees the trajectories of the gradient v exit through B without going back and forward, which provides the existence of some continuous time-keeping function for each point exiting M. We shall elaborate this fact in the next section. We are aware that similar conditions have been introduced in the literature for the construction of relative Morse complex and Morse homology, compare [4], [1] and [8]. Our assumptions are similar to the ones in [4], whereas [1] takes a different model by assuming the flow will never go out of M but travel along the boundary instead.

Remark 1.1.7 B is closed in ∂M, and the assumption **A2** means Γ as the boundary of B is also smooth. In fact, the whole boundary ∂M can be viewed as the union of two closed subsets, namely, the exit set B and the entry set T according to the directions of the trajectories of v, i.e. $\partial M = B \cup T$ and $\Gamma = B \cap T$. One can also consider the case $\Gamma = \emptyset$, in which case we generalise the statements in Milnor [29] by allowing f to be nonconstant on B.

We stick to these boundary assumptions on manifolds with boundary throughout the thesis.

1.2 Homotopy type

Primarily, we want to understand the homotopy type of a manifold with boundary under the evaluation of a Morse function. In this section, we give a geometric account of the problem. The deformation retraction of the manifold essentially comes from the negative gradient flow of the underlying function, similar to the one described in [28].

Notation 1.2.1 Suppose $c \in \mathbb{R}$ is a regular value of f on M, we denote $M^c = f^{-1}(-\infty, c]$ and $B^c = B \cap M^c$.

The new issue here is mainly to describe the exiting process of the negative flow via B, whereas the handle body argument concerning the interior of the manifold will be taken care of in the same fashion as described in [28, Theorem 3.2, page 14].

Theorem 1.2.2 Let M be a manifold with boundary ∂M, and $f : M \to \mathbb{R}$ be a Morse function on M with exit set $B \subset \partial M$ and satisfying conditions **A1**, **A2** and **A3**. Moreover, if f has no critical points in M^c, then B^c is a deformation retract of M^c, $M^c \simeq B^c$.

Proof: Let v^+ be the gradient vector field of f and $\Phi : M^+ \times \mathbb{R} \to M^+$ be the negative flow of $-v^+$. Now because f has no critical point in M^c, for any $x \in M^c$, according to Lemma 1.2.3, there exists a function $\beta : M^c \to \mathbb{R}$ such that

$$\Phi(x, \beta(x)) \in B^c$$

Also by Lemma 1.2.3 below, β is well-defined and continuous, and we define the deformation retraction $r : M^c \times [0,1] \to M^c$ as:

$$r(x,t) = \Phi(x, t\beta(x)),$$

for $x \in M^c$ and $0 \leq t \leq 1$, then r_0 is the identity map and r_1 deformation retracts M^c into B^c. $\square$

Lemma 1.2.3 Suppose f has no critical points on M^c, then there exists a real function $\beta : M^c \to \mathbb{R}$ such that for each $x \in M^c$,

$$\Phi(x, \beta(x)) \in B^c.$$

Proof: Define $\beta : M^c \to \mathbb{R}$ as

$$\beta(x) = \min\{t : x \cdot t \in B\}.$$

We want to show β is well-defined by arguing its existence and uniqueness, and that β is continuous (in fact, in this case, β is smooth).

Choose a point $x \in M^c$, let $\gamma : \mathbb{R} \to M^c$ be the integral curve of $-v^+$, with initial condition $\gamma(0) = x$. We want to show by contradiction that after some finite time $t_x \in \mathbb{R}$, $\gamma(t)$ will be permanently out of M for any $t > t_x$. Now $\gamma'(t) = -v(\gamma(t))$, and

$$(f \cdot \gamma)'(t) = df(\gamma(t) \cdot \gamma'(t) = -df(\gamma(t)(v(\gamma(t)) = -\langle v, v\rangle < 0,$$

according to **A1**.

Then by the compactness of M^c, there exists $K > 0$ such that $(f \cdot \gamma)'(t) = -\langle v(\gamma(t)), v(\gamma(t))\rangle \leq -K < 0$ for $t \in \mathbb{R}$. Suppose on the other hand, for all $t \in \mathbb{R}$, γ can be extended such that $\gamma(t) \in M^c$, i.e. the trajectory lives in M^c forever, then

$$\lim_{t\to\infty} \frac{f \cdot \gamma(0) - f \cdot \gamma(t)}{t} = 0,$$

since there always exists some $L > 0$ such that $f \cdot \gamma(0) - f \cdot \gamma(t) < L$ by the compactness of M^c. This contradicts to the fact that, by Mean Value Theorem, there always exists some s such that

$$-\frac{f \cdot \gamma(0) - f \cdot \gamma(t)}{t} = (f \cdot \gamma)'(s) \leq -K,$$

for any interval $[0, t]$. Therefore, there exists $t \in \mathbb{R}$, such that $\gamma(t) \notin M^c$, according to the Escape Lemma in [24, p.446, Lemma 17.10]. Now by Intermediate Value Theorem, we conclude that there exists $t_0 \in \mathbb{R}$, such that $\gamma(t_0) \in B$.

Now we want to show that the map β is continuous. Consider the negative gradient flow Φ of $-v^+$, observe that for any point $x \in \operatorname{Int} M^c$ there exists $t_x \in \mathbb{R}$

and $b_x \in \operatorname{Int} B$ such that $x \cdot t_x = b_x$. This is true as the assumptions and the choice of B exclude the possibility that any points in the interior would reach Γ. Namely, $\Gamma = T \cap B$ is the intersection of the entry set and the exit set, and we know from assumption **A3** that $\frac{\partial f}{\partial s}(x, 0, 0) > 0$ for $(x, 0, 0) \in \Gamma \times [-1, 1] \times [-1, 1]$. Hence with the product structure of the Riemannian metric on $\partial M \times [-1, 1]$, $\gamma_x(t) \in T \times [-1, 0]$ for any $t < 0$ and $\gamma_x'(0) \in T_x B$, where $\gamma_x : \mathbb{R} \to M^+$ is the integral curve with initial position $\gamma(0) = x$. This insists that any point $(x, 0, 0) \in \Gamma \times [-1, 1] \times [-1, 1]$ has to come from $M^+ - M$ outside of M and flows towards B. The following figure demonstrates this fact graphically.

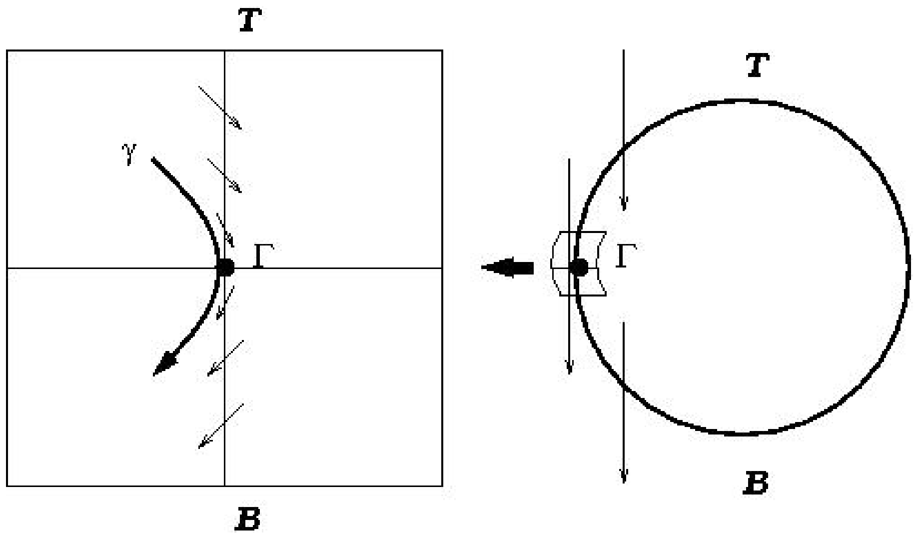

Figure 1.1: Gradient vector field and flow at the cubical neighbourhood of Γ

Now for $x \in \Gamma$, β is continuous near Γ by observing that for any $\epsilon > 0$, $x \cdot \epsilon \notin M$, and because the continuity of the flow, there exists a sufficiently small open neighbourhoods U_x of x and δ_x such that for any $y \in U_x$ there exists $t \in [\epsilon - \delta_x, \epsilon + \delta_x]$ with $y \cdot t \notin M$.

Therefore, we are left to show the continuity of $\beta : M^c \to \mathbb{R}$ for $x \notin \Gamma$.

Choose small open neighbourhoods U of x in M^c and V of b_x in M^+ and an open interval $I \subset \mathbb{R}$ containing t_x, such that $\Phi(U \times I) \subset V$. In local coordinates $x = (x_1, \ldots, x_m)$ and $b_x = (y_1, \ldots, y_{m-1}, 0)$ of $\mathbb{R}^m$ in U and V, respectively.

Now let $\pi : \mathbb{R}^m \to \mathbb{R}$ be the projection of $\mathbb{R}^m$ to the last coordinate, and compose function $h = \pi\Phi : U \times I \to \mathbb{R}$, then $h(x, t_x) = 0$ and

$$\frac{\partial h}{\partial t} = \frac{\partial(\pi\Phi)}{\partial t} = \mathrm{d}(\pi\Phi)(\frac{\partial}{\partial t}) = \mathrm{d}\pi\left(\mathrm{d}\Phi\frac{\partial}{\partial t}\right) = \mathrm{d}\pi\frac{\partial\Phi}{\partial t}.$$

Now for any $(x,t) \in M^+ \times \mathbb{R}$, $\frac{\partial\Phi}{\partial t}(x,t) = -v(\Phi(x,t)$, therefore for $(x,t_x) \in M^c \times \mathbb{R}$ with $x \cdot t_x = b_x \in \operatorname{Int} B$:

$$\frac{\partial h}{\partial t}(x,t_x) = \mathrm{d}\pi\big(-v(\Phi(x,t_x))\big) < 0.$$

by assumption **A2** and the choice of B, where $-\dfrac{\partial f}{\partial t} < 0$.

Therefore, by Implicit Function Theorem, there exists open neighbourhoods $W \subset U$ of x_0 and $L \subset I$ of t_0 and a smooth function $g : W \to L$ such that for each $(x,t) \in W \times L$,

$$h(x,t) = 0 \ (i.e.,\ x \cdot t \in \operatorname{Int} B) \Longleftrightarrow g(x) = t$$

Therefore, the lemma is true. □

Lemma 1.2.4 Suppose $f^{-1}[a,b]$ contains no critical points of f for $-\infty < a < b < \infty$, then $B^b \cup M^a$ is a deformation retract of M^b, $M^b \simeq B^b \cup M^a$.

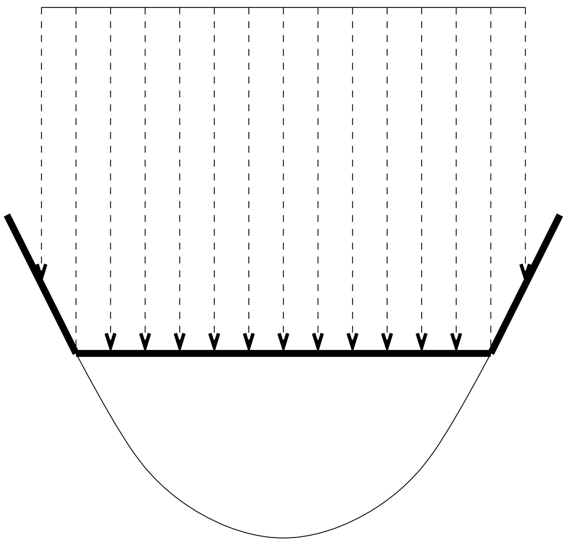

Figure 1.2: Deformation with corners

Proof: Let Φ be the flow of $-v$, then we define E and F as subsets of $f^{-1}[a,b]$ as

$$E = \{x \in f^{-1}[a,b] : \text{ there exists } t \text{ with } x \cdot t \in B \cap f^{-1}[a,b]\}$$

and

$$F = \{x \in f^{-1}[a,b] : x \cdot t \in f^{-1}(\{a\}) \text{ for some t, and } x \cdot t' \notin B \text{ for all } t' < t\}.$$

By Proposition 1.2.5 below, we know that both E and F are closed subsets of $f^{-1}[a,b]$. Then according to Lemma 1.2.3 we have a real continuous function β on E, and by the construction of F there exists real continuous function $\alpha : F \to \mathbb{R}$ such that for any $x \in F$, $x \cdot \alpha(x) \in f^{-1}(\{a\})$. Clearly α and β agree on $E \cap F$, and since they are both closed, we can paste the two functions together, then we obtain a continuous function $h : f^{-1}[a,b] \to \mathbb{R}$, with $h|_E = \beta$ and $h|_F = \alpha$, such that for any $x \in f^{-1}[a,b]$,

$$\Phi(x, h(x)) \in B^b \cup M^a.$$

Define the deformation retraction $r : M^b \times I \to M^b$ as

$$r(x,t) = \Phi\big(x, th(x)\big)$$

It is clear that r_0 is the identity map and r_1 deformation retracts M^a into $M^b \cup B^a$.
□

Now we are left to show both E and F are closed.

Proposition 1.2.5 Both E and F are closed.

Proof: Let $x \in f^{-1}[a,b] \setminus E$, then $x \cdot t_x \in (B^b - B \cap f^{-1}[a,b])$ for some $t_x \in \mathbb{R}$. Since $B^b - B \cap f^{-1}[a,b]$ is open, we can always find a small open neighbourhood of $x \cdot t_x$ disjoint from $B \cap f^{-1}[a,b]$, and using the continuity of Φ, there exists a small open neighbourhood of x disjoint of E. Hence, E is closed.

There is a similar argument for F. Namely, let $x \in f^{-1}[a,b] \setminus F$, i.e. there exists no $t \in \mathbb{R}$ such that $x \cdot t \in f^{-1}(\{a\})$, which means $x \cdot t_x \in \operatorname{Int}(B \cap f^{-1}[a,b])$ for some $t_x \in \mathbb{R}$, and here $\operatorname{Int} B \cap f^{-1}[a,b]$ is open, we can always find some open neighbourhood of $x \cdot t_x$ disjoint from $f^{-1}(\{a\})$, and again using the continuity of Φ, there exists an open neighbourhood of x in $f^{-1}[a,b] \setminus F$, so F is closed. $\square$

Lemma 1.2.3 and Lemma 1.2.4 and their proofs together actually can tell us more of the points flowing out via B:

Corollary 1.2.6 If $x \in M$ and $x \cdot t_x \in B$ for some $t_x \in \mathbb{R}$, then there exists a continuous function $\beta_x : U_x \to \mathbb{R}$ on a neighbourhood U_x of x, such that for a point $y \in U_x$, the function maps y to a real number $\beta_x(y)$ with $y \cdot \beta_x(y) \in B$.

Remark 1.2.7 Corollary 1.2.6 describes the local behaviour of the points exiting through B, its usage will be exhibited in details when we study the *Lusternik-Schnirelman category with respect to a closed 1-form* in chapter 6.

Lastly, we need a statement about the homotopy type of an interval of f^{-1} containing critical points of f, e.g. in Milnor's Morse Theory [28]:

Proposition 1.2.8 Let p be a critical point of $f : M \to \mathbb{R}$ and $f(p) = c$, choose $\epsilon > 0$ small enough such that $f^{-1}[c+\epsilon, c-\epsilon]$ has the only critical point p. Then $M^{c+\epsilon} \simeq B^{c+\epsilon} \cup M^{c-\epsilon} \cup e^{\lambda}$, where λ is the index of p. $\square$

Together, the following theorem states the homotopy type of a manifold with boundary on which the assigned Morse function satisfying assumptions **A1**, **A2** and **A3**:

Theorem 1.2.9 Let M be a manifold with boundary ∂M, $f : M \to \mathbb{R}$ be a smooth Morse function and B be the exit set of f in ∂M, suppose f satisfies assumptions **A1**, **A2** and **A3**, then (M, B) has the homotopy type of a relative CW-complex, with one cell of dimension λ attached to $(M, B)^{(\lambda-1)}$ for each critical point of index λ, where $(M, B)^{(k)}$ is the k-skeleton of M relative to B. $\square$

1.3 Chain complex

After the intuitive geometric portrayal in the preceding section, we want to move to some more abstract yet computable level of the issue. In this section, we construct the Morse complex for a manifold with boundary and show its chain homotopy equivalence with the simplicial chain complex. The terminology introduced in this section will be extensively used in the rest of the thesis. The essential techniques are first given in [38].

With a fixed Riemannian metric, let Φ be the negative gradient flow on M for a given Morse function f, we shall start with the following core notions for the construction of the chain complex:

Definition 1.3.1 (Stable and unstable manifolds of Φ) Let p be a nondegenerate critical point of f, then the *stable manifold* $W^s(p,v)$ at p is defined as:

$$W^s(p,v) = \{x \in M : x \cdot t \to p \text{ as } t \to -\infty\},$$

and similarly, the *unstable manifold* $W^u(p,v)$:

$$W^u(p,v) = \{x \in M : x \cdot t \to p \text{ as } t \to +\infty\},$$

Where $x \cdot t = \Phi(X,t)$.

Suppose the critical point p has index i, then the stable manifold $W^s(p,v)$ is a i-dimensional immersed submanifold of M, and the unstable manifold $W^u(p,v)$ is a $(m-i)$-dimensional immersed submanifold of M. For a detailed account regarding the immersion of the stable and unstable manifolds we refer to [3, Chapter 6]. Furthermore, we need the following *transversality condition* on gradient vector field v of f:

Definition 1.3.2 We say a gradient vector field v of f is *transverse* if for any critical points p, q with $\operatorname{ind} q \leq \operatorname{ind} p + 1$, the unstable manifold $W^u(p,v)$ of p intersects with the stable manifold $W^s(q,v)$ of q transversely.

Remark 1.3.3 The transversality condition of v is a generic property, according to Smale [39]. Because of the compactness of our manifold, by the techniques introduced in [29, Chapter 5, Theorem 5.2], we can and will always have it by some small perturbation of any given gradient vector field. However, when it comes to circle-valued functions and closed 1-forms in the coming chapters, some Baire type

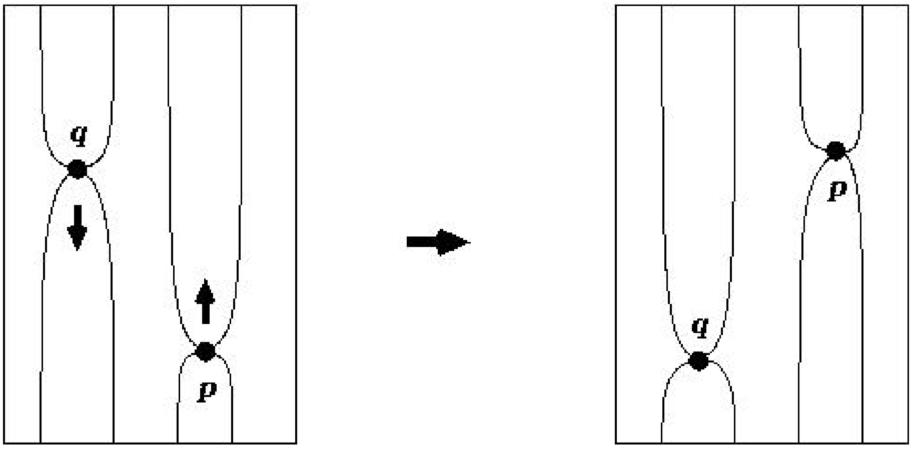

Figure 1.3: Self-indexing rearrangement

argument will be needed. Moreover, the transversality property is obviously preserved when switching to the negative gradient.

Now if v is transverse, [29] tells us that the function f can be perturbed locally and enjoys the self-indexing property, namely, the critical points of f can be rearranged according to their indices so that for $p, q \in \operatorname{Crit} f$ with $\operatorname{ind}(p) > \operatorname{ind}(q)$, then $f(p) > f(q)$. See Figure 1.4 above.

However, a potential problem can arise if the space between the exit set and entry set of the boundary is too narrow as depicted in the first picture of Figure 1.4. We need to remove such obstruction by pushing up the entry set so that we have enough room to bring the critical points with higher index above the lower ones. See the second picture of Figure 1.3 below. Note that this may change the function f on T, but the cellular structure remains the same.

Then we can construct a filtration of M as:

$$B = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M,$$

where $M_i - \operatorname{Int} M_{i-1}$ contains only the critical points of index i.

Consider the relative homology $H_i(M_i, M_{i-1})$ of the filtration, according to Proposition 1.2.8,

$$H_i(M_i, M_{i-1}) \simeq H_i(M_{i-1} \cup e_1^i \cup \cdots \cup e_n^i, M_{i-1}) = \mathbb{Z}^n,$$

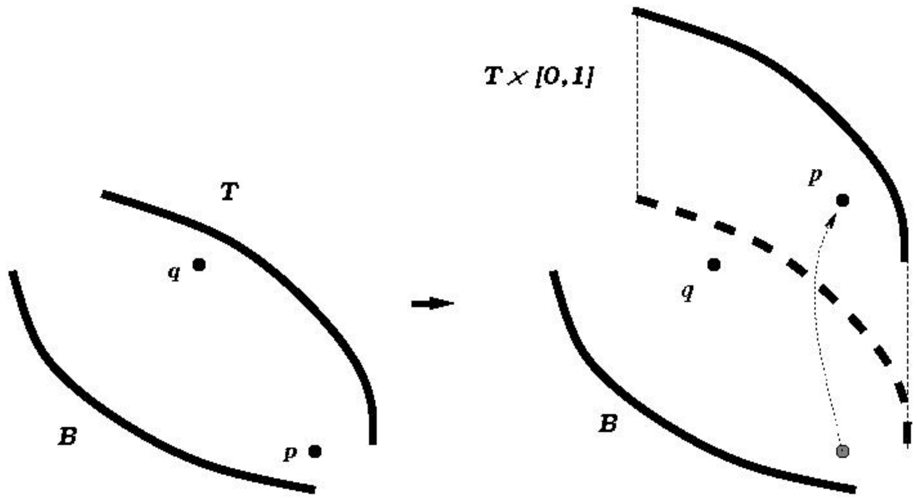

Figure 1.4: Pushing up the entry set T

where e_k^i is a stable disc containing a critical point of index i and n is the number of such critical points in M.

So we define the ith chain group $C_i(M, B, f, v) = H_i(M_i, M_{i-1})$, a free abelian group finitely generated by critical points of index i, and boundary map $\partial : C_i(M, B, f, v) \to C_{i-1}(M, B, f, v)$ to be the connecting homomorphism of the triple (M_i, M_{i-1}, M_{i-2}). Next, we describe the geometric interpretation of this boundary map:

Definition 1.3.4 (Orientation of the manifolds) Choose an orientation of $W^s(p, v)$ and an orientation of the normal bundle of $W^u(p, v)$ such that its projection to p is consistent with the orientation of $W^s(p, v)$ at p. Also we call the orientation of the normal bundle of $W^u(p, v)$ *coorientation.* Consider the trajectory γ of $-v$ between p, q where $\operatorname{ind} p = i = \operatorname{ind} q + 1$, let $X_1, \cdots, X_i \in T_{\gamma(t)}M$ represent the coorientation of $W^u(q, v)$, then if the projection of $\gamma'(t), X_1, \cdots, X_i$ onto $T_{\gamma(t)}W^s(p, v)$ is consistent with the orientation of $W^s(p, v)$, denote $\epsilon(\gamma) = 1$ otherwise $\epsilon(\gamma) = -1$.

Definition 1.3.5 (Incidence number between critical points) Let $p : \tilde{M} \to M$ be a regular covering space with covering transformation group G, also denote the lift of the Morse function f and its gradient v as $\tilde{f}$ and $\tilde{v}$ respectively, then we have the covering chain complex $C_*(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$ as:

$$C_i(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) = H_i(\tilde{M}_i, \tilde{M}_{i-1}),$$

and it can be seen as a $\mathbb{Z}G$ module as there is a group action of G on $\tilde{M}$.

Choose for every critical point p of f exactly one lift $\tilde{p}$ in $\tilde{M}$, then we define the *incidence number* $[\tilde{p} : g\tilde{q}] \in \mathbb{Z}G$ as

$$[\tilde{p} : g\tilde{q}] = \sum \epsilon(\gamma) \cdot g,$$

where $\operatorname{ind} \tilde{p} = \operatorname{ind} \tilde{q} + 1$, $g \in G$ and the sum is taken over the finite set of trajectories between $\tilde{p}$ and $g\tilde{q}$.

Now for each pair p and q, regardless the choice of lift, there are only finite components in the summation $\sum_{g \in G} = [\tilde{p}, g\tilde{q}]\tilde{q}$, as the number of trajectories between p and q in the base is finite.

Definition 1.3.6 (Boundary map) After choosing a lift $\tilde{p}$ for each critical point $p \in \operatorname{Crit} f$ in the covering space, we adapt the results in [29, pp.86-89] then the *boundary map* $\partial : C_{i+1}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) \to C_i(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$ can be identified as:

$$\partial(\tilde{p}) = \sum_{q \in \operatorname{Crit}_i f} \sum_{g \in G} [\tilde{p} : g\tilde{q}]\tilde{q}$$

where $[\tilde{p} : g\tilde{q}]$ is the *incidence number* of $\tilde{p}$ and $g\tilde{q}$ by counting the trajectories from $\tilde{p}$ to $g\tilde{q}$ with respect to orientation. Note it is well-defined as $\sum_{g \in G} [\tilde{p} : g\tilde{q}] \in \mathbb{Z}G$ and $\partial^2 = 0$.

Now we successfully relate the chain complex of the filtration to the chain complex of the CW-complex produced in the end of section 1, and we state it in the following definition:

Definition 1.3.7 The *Relative Morse-Smale complex* of a Morse function f on a manifold with boundary is given by:

$$C_i^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) = \bigoplus_{p \in \operatorname{crit}_i(f)} \mathbb{Z}G$$

as a $\mathbb{Z}G$-module with generators corresponding to the critical points of the function f.

Before moving on, we want to modify the filtration of the manifold to prepare the construction of the chain homotopy map between the simplicial chain complex and the Morse complex, this idea first appeared in [38] and is essentially useful for our construction of the chain complexes in the rest of the thesis.

Definition 1.3.8 (Modification of the filtration) To begin with, we need a map, say $\Theta : M \times \mathbb{R} \to M$ similar to a complete flow which agrees with the gradient flow of $-v$ except in a sufficiently small neighbourhood of ∂M. Recall the bigger manifold M^+ in the beginning of Section 1, the gradient vector field v^+ of $f : M^+ \to \mathbb{R}$ vanishes at $\partial M \times \{-1\}$. This vector field generates a global flow on M^+. Now we retract M^+ back to M and the flow is the Θ we wanted, in particular, notice that $\Theta(x,t) = x$ for $(x,t) \in \partial M \times \{-1\} \times [0,\infty)$. Denote $\Theta_t(x) = \Theta(x,t)$ and define:

$$M_i^t = \begin{cases} \Theta_t(M_i) & \text{if } t \geq 0 \\ \bigcup_{0 \geq s \geq t} \Theta_s(M_i) & \text{if } t < 0 \end{cases}$$

Now since each Θ_t is a diffeomorphism of M, $\{M_i^t\}_{i=-1}^n$ is also a filtration of M, and $H_i(\tilde{M}_i^t, \tilde{M}_{i-1}^t) \cong H_i(\tilde{M}_i, \tilde{M}_{i-1})$ induced by inclusion.

Note that for $t > s$ we have $M_i^t \subset M_i^s$, and for very large t, M_i^t contains mainly B and the stable manifolds $W^s(p,v)$ with ind $p \leq i$, and for very negative t, M_i^t contains mainly the complement of the union of unstable manifolds $W^u(p,v)$ with ind $p \geq i+1$. And this actually suggests a direct system $j_{st} : H_*(\tilde{M}_i^t, \tilde{M}_{i-1}^t) \to H_*(\tilde{M}_i^s, \tilde{M}_{i-1}^s)$ where each j_{st} is an isomorphism and commutes with the boundary maps:

$$\begin{tikzcd} H_i(\tilde{M}_i^t, \tilde{M}_{i-1}^t) \arrow[r, "\partial_i"] \arrow[d, "j_{st}"'] & H_{i-1}(\tilde{M}_{i-1}^t, \tilde{M}_{i-2}^t) \arrow[d, "j_{st}"] \\ H_i(\tilde{M}_i^s, \tilde{M}_{i-1}^s) \arrow[r, "\partial_i"] & H_{i-1}(\tilde{M}_{i-1}^t, \tilde{M}_{i-2}^t) \end{tikzcd}$$

So let us define

$$C_i = M - \bigcup_{p, \text{ind}\, p \geq i+1} W^u(p,v),$$

then

$$H_i(\tilde{C}_1, \tilde{C}_{i-1}) = \lim_{\to_{j_{st}}} H_i(\tilde{M}_i^t, \tilde{M}_{i-1}^t) \cong H_i(\tilde{M}_i, \tilde{M}_{i-1}) \simeq C_i^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$$

Remark 1.3.9 In the language of spectral sequence, $H_i(\tilde{C}_i, \tilde{C}_{i-1})$ can be seen as the E^1 term of the spectral sequence, derived from the above filtration:

$$B \subset C_1 \subset \cdots \subset C_n = M.$$

Definition 1.3.10 A triangulation Δ of M is said to be *adjusted to* v the gradient vector field of f if the intersection of every i-simplex of Δ, σ^i, with the unstable manifolds $W^u(p, v)$ of v is transverse, for all p with ind $p \geq i$. Such a triangulation is called *an adjusted triangulation to* v. We use the notation Δ for a triangulation. We denote the simplicial chain complex of a triangulated manifold M as

$$C_*^{\Delta}(M, \partial M)$$

The existence of adjusted triangulation is plenty, and [38] outline an inductive construction starting from the 0-skeleton, here we only need to care about the exit B, but since assumptions $\mathbf{A1}, \mathbf{A2}$ and $\mathbf{A3}$ ensure the transversality of the flow to B, the construction is essentially provided in [38].

For the convenience of the reader, we sketch here an inductive construction of such triangulation in a general setting:

Given a random triangulation Δ, we show by a small perturbation, namely, a diffeomorphism $\psi : M \to M$ isotopic to the identity, such that $\psi\Delta$ is adjusted to v. Suppose we begin with B, since the assumptions $\mathbf{A1}, \mathbf{A2}$ and $\mathbf{A3}$ have guaranteed the transverse intersection of the exiting flow with B, the original triangulation will do for B. Now for the relative 0-skeleton $(M, B)^{(0)}$, thanks to the compactness of M, $\psi_0 : (M, B)^{(0)} \to (M, B)^{(0)}$ can be chosen by hand such that all the finitely many 0-simplices composing with ψ_0 are transverse to a finite number of all the unstable the unstable manifolds. Suppose this has been done up to $(k-1)$-skeleton $(M, B)^{(k-1)}$, so that every i-simplex of $\psi_{k-1}\Delta$ with $i \leq k-1$ is transverse to unstable manifolds of critical points with index bigger or equal to $k-1$. Then the transversality on k skeleton $(M, B)^{(k)}$ can be done inductively: $\psi_k\Delta$ is already adjusted near the boundary of each k-simplex, so we leave a small neighbourhood of $(k-1)$-skeleton unchanged under ψ_k, and modify the rest of each simplex in a smooth fashion so then $\psi_k\Delta$ is readily adjusted to v as intended.

Now we are ready to define a chain map between the relative simplicial chain complex of the manifold $C_*^{\Delta}(\tilde{M}, \partial\tilde{M})$ and its relative Morse-Smale complex $C_*^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$ induced by the Morse function f:

Definition 1.3.11 (Chain map φ_v) Denote the i-skeleton of the triangulated manifold as $M^{(i)}$, and assuming an adjusted triangulation Δ we have inclusion $M^{(i)} \subset C_i$, because $\sigma \cap W^u(p, v) = \emptyset$ for i-simplex σ and critical point p with $\mathrm{ind}\,(p) \geq i + 1$ as transversality implies $\dim(\sigma \cap W^u(p, v)) = -1$. Now view the simplicial complex as a cellular chain complex which shares the same homology, namely, the ith chain $C_i^{\Delta}(M, B) = H_i(M^{(i)}, M^{(i-1)})$, the above inclusion induces a map on this homology: $H_i(M^{(i)}, M^{(i-1)}) \to H_i(C_i, C_{i-1})$ which commutes with ∂ and hence

$$\varphi_v : C_*^{\Delta}(M, B) \to C_*^{\mathrm{MS}}(M, B, f, v)$$

is well-defined.

Now with $\dim \sigma = \mathrm{ind}\, p$, the intersection $\sigma \cap W^u(p, v)$ is a finite set of points by the transversality condition as $\dim(\sigma \cap W^u(p, v)) = 0$. By fixing an orientation of the triangulation Δ, these points actually function in a book-keeping role on how each free generator σ of $H_i(M^{(i)}, M^{(i-1)})$ is embedded in C_i. To be precise, consider the case of only one critical point p of index i, we have the isomorphism of the homology

$$C_i^{\mathrm{MS}}(M, B, f, v) = H_i(C_i, C_{i-1}) \simeq H_i(M_{i-1} \cup D(p) \cup W^u(p, v), M_{i-1} \cup (e^i(p) - p))$$

where

$$e^i(p) = W^s(p, v) \cap (M_i - \mathrm{Int}\,(M_{i-1}))$$

is the i-disc of critical point p. In the case of the latter homology, the image $\varphi_v(\sigma)$ of σ matters only in the intersection with the unstable manifold $W^u(p, v)$. Now $W^u(p, v)$ is shrinking along the modified flow Θ in Definition 1.3.7, $\Theta(\varphi_v(\sigma))$ arrives next to the i-disc $e^i(p)$ of p in finite time, So depending on the orientation, each intersection point of $\sigma \cap W^u(p, v)$ corresponds to a copy of $e^i(p)$ up to sign. Therefore we lift everything to the covering and use the same notation φ_v , then we obtain a similar *incidence number* $[\sigma : p] \in \mathbb{Z}G$ after comparing σ with the coorientation of $W^u(p, v)$:

$$\varphi_v(\tilde{\sigma}) = \sum_{\substack{p\in \mathrm{Crit}\, f \\ \mathrm{ind}\, p=\dim \sigma}} \sum_{g\in G} [\tilde{\sigma} : g\tilde{p}]\tilde{p}.$$

With the standard boundary map $\partial^\Delta : C^\Delta_*(\tilde{M}, \tilde{B}) \to C^\Delta_{*-1}(\tilde{M}, \tilde{B})$, it is clear the following diagram commutes:

$$\begin{tikzcd}
C^\Delta_*(\tilde{M}, \tilde{B}) \arrow[r, "\varphi_v"] \arrow[d, "\partial^\Delta"] & C^{\mathrm{MS}}_*(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) \arrow[d, "\partial"] \\
C^\Delta_{*-1}(\tilde{M}, \tilde{B}) \arrow[r, "\varphi_v"] & C^{\mathrm{MS}}_*(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})
\end{tikzcd}$$

Now we are ready to state the homotopy equivalence of the two relative chain complexes based on the techniques in [38, Theorem 2.5]:

Theorem 1.3.12 Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold M with boundary ∂M satisfying boundary assumptions **A1**, **A2** and **A3**, and v be a transverse gradient vector field of f, also given an adjusted triangulation Δ, then the chain map $\varphi_v : C^\Delta_*(\tilde{M}, \tilde{B}) \to C^{\mathrm{MS}}_*(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$ is a chain homotopy equivalence.

Proof: Observe first that if Δ' is a subdivision of Δ, Δ' adjusted to v induces Δ adjusted to v, and furthermore, we have the following commutative diagram:

$$\begin{tikzcd}
C^\Delta_*(\tilde{M}, \tilde{B}) \arrow[rr, "\text{subdivision}"] \arrow[dr, "\varphi_v"'] & & C^{\Delta'}_*(\tilde{M}, \tilde{B}) \arrow[dl, "\varphi_v"] \\
& C^{\mathrm{MS}}_*(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) &
\end{tikzcd}$$

Therefore, according to Munkres [31] it is good enough to show the theorem for a special smooth triangulation Δ. Now consider again the filtration in the beginning of the section, we subdivide Δ if necessary so that each M_i is a subcomplex for $-1 \le i \le n$, and since subdivision preserves the dimension of each subcomplex, M_i is of dimension i as a subcomplex of the i-th skeleton $M^{(i)}$, $M_i \subset M^{(i)}$, Moreover, each stable disc $e^i(p) = W^s(p, v) \cap (M_i - \mathrm{Int}\,(M_{i-1}))$ containing the critical point p with index i is a subcomplex.

Now we set $C^{(k)}_i = C^\Delta_i(\tilde{M}_k, \tilde{B})$ and $D^{(k)}_i = C^{MS}_i(\tilde{M}_k, \tilde{B})$.

Also the chain map $\varphi_v : C^\Delta(\tilde{M}, \tilde{B}) \to C^{\mathrm{MS}}(\tilde{M}, \tilde{B})$ induces $\varphi^{(k)} : C^{(k)} \to D^{(k)}$ and $\varphi^{(k,k-1)} : C^{(k)}/C^{(k-1)} \to D^{(k)}/D^{(k-1)}$.

Now notice that $C^{(k)}/C^{(k-1)} = C_*^{\Delta}(\tilde{M}_k, \tilde{M}_{k-1})$ and we have the following isomorphisms for $D^{(k)}/D^{(k-1)}$:

$$D_*^{(k)}/D_*^{(k-1)} = C_k^{\mathrm{MS}}(\tilde{M}_k, \tilde{M}_{k-1}) = H_k(\tilde{M}_k, \tilde{M}_{k-1}) \cong H_k(\tilde{M}_{k-1} \cup \bigcup_{p \in \mathrm{Crit}_k(\tilde{f})} e^k(p), \tilde{M}_{k-1}),$$

i.e. the quotient is nontrivial only at degree k.

Considering $C^{(k)}/C^{(k-1)} = C_*^{\Delta}(\tilde{M}_k, \tilde{M}_{k-1})$, let $\sigma \in C_k^{\Delta}(\tilde{M}_k, \tilde{M}_{k-1})$, because of the choice of subdivision and transverse intersection of simplices with the unstable manifolds of M, if σ doesn't contain any critical points of index k, then $\varphi(\sigma) = 0$ by the definition of φ above. Therefore the only nontrivial preimages of φ are those contain critical points of f and they are sent to the elements of $H_k(\tilde{M}_{k-1} \cup \bigcup_{p \in \mathrm{Crit}_k(\tilde{f})} e^k(p), \tilde{M}_{k-1})$ corresponding to the critical points they contain: if $p \in \sigma \subset \tilde{M}_k - \tilde{M}_{k-1}$ then

$$\varphi(\sigma) = \pm e^k(p),$$

where the signs depend on the orientation of σ and $e^k(p)$.

Now since $e^k(p) = W^s(p, v) \cap (M_k - \mathrm{Int}\,(M_{-1}k))$ is a subcomplex, i.e. can be written as a sum of simplices: $e^k(p) = \sum_{\sigma \subset e^k(p)} \sigma$, moreover, it is a k cycle in $C^{(k)}/C^{(k-1)}$, therefore,

$$\varphi(\sum_{\sigma \subset e^k(p)} \sigma) = \varphi(\sigma_p) = \pm e^k(p).$$

where σ_p is the simplex containing critical point p. So such one-to-one correspondence $\varphi^{(k,k-1)} : C^{(k)}/C^{(k-1)} \to D^{(k)}/D^{(k-1)}$ induces isomorphisms in homology:

$$\varphi_*^{(k,k-1)} : H_*(C^{(k)}/C^{(k-1)}) \to H_*(D^{(k)}/D^{(k-1)}).$$

Now we use induction and Five Lemma for the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^{(k-1)} \arrow[r] \arrow[d, "\varphi^{(k-1)}"] & C^{(k)} \arrow[r] \arrow[d, "\varphi(k)"] & C^{(k)}/C^{(k-1)} \arrow[r] \arrow[d, "\varphi^{(k,k-1)}"] & 0 \\
0 \arrow[r] & D^{(k-1)} \arrow[r] & D^{(k)} \arrow[r] & D^{(k)}/D^{(k-1)} \arrow[r] & 0
\end{tikzcd}$$

There $\varphi^{(k)}$ is chain homotopy equivalence for each k, hence $\varphi_v : C^{\Delta}(\tilde{M}, \tilde{B}) \simeq C^{\mathrm{MS}}(\tilde{M}, \tilde{B})$. $\square$

We want to end this section by introducing another chain map, between two Morse chain complexes induced by different gradients. It is useful when we want to compare two Morse chain complexes of a manifold.

Definition 1.3.13 For a manifold M with boundary ∂M, let $f, g : M \to \mathbb{R}$ be two Morse functions on M with the same exit set $B \subset \partial M$, and we also have v_f and v_g the gradients of f and g, respectively. Then we have two relative Morse complexes $C_*^{\mathrm{MS}}(M, B, f, v_f)$ and $C_*^{\mathrm{MS}}(M, B, g, v_g)$. Now follow the construction in [29, Lemma 5.3], we have a map $\Psi : M \to M$ isotopic to identity so that $\Psi(W^s(q, v_f))$ intersects $W^u(p, v_g)$ transversely for each $q \in \operatorname{Crit} f$ and $p \in \operatorname{Crit} g$. Since M is compact, we can necessarily repeat Lemma 5.3 in [29] inductively in finitely many times to ensure the existence of Ψ. Then the critical points of (f, v_f) introduce the filtration of M as

$$B = M_{-1}(v_f) \subset M_0(v_f) \in \cdots \in M_m(v_f) = M,$$

where each $M_i(v_f)$ contains the critical points of f of index i only, similarly we have a filtration of M according to (g, v_g). Now we shrink each $M_i(v_f)$ by the flow Θ defined in Definition 1.3.8, so that it is contained in $C_i(v_g)$. Recall $C_i(v_g)$ is the direct limit of the direct system induced by the flow of v_g. So we have a chain map φ_{v_f, v_g} between the two complexes:

$$\varphi_{v_f, v_g} : C_*^{\mathrm{MS}}(M, B, f, v_f) \to C_*^{\mathrm{MS}}(M, B, g, v_g),$$

induced by inclusion. Namely, after fixing orientation, we have incidence number $[q : p]$ for each $q \in \operatorname{Crit} f$ and $p \in \operatorname{Crit} g$ with same index by reading off the intersection points of $\Psi(W^s(q, v_f)) \cap W^u(p, v_g)$, so we can write down φ_{v_f, v_g} as

$$\varphi_{v_f, v_g}(q) = \sum_{p \in \operatorname{Crit}_{\operatorname{ind}(q)} g} [q : p] p.$$

1.4 Homology and Morse inequalities

According to the definition of the Morse chain complex $C_*^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$, $C_i^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$ is free and finitely generated by elements corresponding to the critical points of f, having index i for each $i = 0, \ldots, n$. Using standard arguments in the proof of the Euler-Poincaré theorem, we obtain the Morse inequalities statement for relative Morse complex:

Theorem 1.4.1 Denote $\beta_i = \operatorname{rank} H_i(M, B)$ as the rank of $H_i(M, B)$ and

$$c_i = \operatorname{rank} C_i^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) = |\{\operatorname{Crit}_i f\}|$$

as the rank of $C_i^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$ or the cardinality of the set $\operatorname{Crit}_i f$, then we have the following equation for a non-negative polynomial $R(t)$:

$$\sum_{i=0}^{m} t^i c_i = \sum_{i=0}^{m} t^i \beta_i + (1+t)R(t),$$

where m is the dimension of the manifold M.

Proof: This is equivalent to showing

$$\sum_{i=0}^{k} (-1)^{k-i} c_i \geq \sum_{i=0}^{k} (-1)^{k-i} \beta_i$$

for each $k = 0, \ldots, n$.

Now let $z_i = \operatorname{rank} \ker(\partial_i)$ and $b_i = \operatorname{rank} \operatorname{Im}(\partial_{i+1})$, then by the following short exact sequences:

$$0 \to \operatorname{Im}(\partial_{i+1}) \to \ker(\partial_i) \to H_i(M, B) \to 0$$

and

$$0 \to \ker(\partial_i) \to C_i^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) \to \operatorname{Im}(\partial_i) \to 0,$$

we have

$$z_i = \beta_i + b_i$$

and

$$c_i = z_i + b_{i-1},$$

since $H_*(M, B) \cong H_*(C^{\mathrm{MS}}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}))$ by the homotopy equivalence in the previous section. Then we have

$$c_i = \beta_i + b_i + b_{i-1},$$

and take the alternating sum of it, we get the inequalities stated in the beginning of the proof. □

Chapter 2

Circle-valued functions

We apply the techniques developed in the previous chapter for the real case to a circle-valued function, mainly following the treatment in [38]. Namely, we study the critical points of a circle-valued function by lifting it up to the covering space. In the total space, when restricted to a piece of finite copies of the base space, the problem is reduced to the real case which is studied in the preceding chapter, and by taking inverse limit, an analogous chain homotopy equivalence is obtained. However, in order to demonstrate a clearer understanding of the essential mechanism, we shall first work on the minimal regular covering.

2.1 The minimal covering space

Suppose we have a compact manifold M with boundary ∂M as before, and assign a circle-valued function $f : M \to S^1$ on M. Here we can parametrize S^1 by the map $t \to e^{\pi i t}$ from the reals to S^1, and notice that locally f is real hence we call f *Morse* if all the critical points of f are nondegenerate in local coordinates, and we also assign an index to each critical point accordingly.

To study a genuinely interesting circle-valued function on a manifold with boundary, we may first have some basic assumptions on the fundamental group homomorphism $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$. Firstly let f_* be nonzero, for if $f_* = 0$ we can find a lift of the map $f : M \to S^1$ to the real, which has been studied in the previous sections; further more, we assume f_* to be surjective, otherwise, we can find a lift $\bar{f}$ of f to a finite covering $S^1 \to S^1$ which induces an epimorphism $\bar{f}_*$ on the fundamental groups.

Denote $G = \pi_1(M)$ to be the fundamental group of M, and let $\bar{\rho} : \bar{M} \to M$ be the minimal cyclic covering of M with covering transformation group $\mathbb{Z} = G/\ker(f_*)$. Then we can lift $f \circ \bar{\rho}$ to a real function $\bar{f} : \bar{M} \to \mathbb{R}$ with the assumption that zero is a regular value.

Having a real Morse function $\bar{f} : \bar{M} \to \mathbb{R}$ on the covering space, we can repeat the construction detailed in Chapter 1, first enlarge the manifold M and fix a collaring $\partial M \times [-1, 1]$ for ∂M, then lift it up to the covering, we define the exit set $\bar{B}$ to be the subset of boundary $\partial \bar{M}$ with non-positive partial derivatives along the normal coordinates of $\partial \bar{M} \times [-1, 1]$; On the other hand, partial derivatives $\dfrac{\partial \bar{f}}{\partial t}$ is equivariant under the group transformation on the covering, i.e.

$$\frac{\partial \bar{f}}{\partial t}(gx) = \frac{\partial \bar{f}}{\partial t}(x)$$

for $g \in \mathbb{Z}$. Therefore, $\dfrac{\partial \bar{f}}{\partial t}$ can be seen as a real function on the base manifold M and the assumptions **A1**, **A2** and **A3** are defined as in the previous chapter. We also have exit B defined by $\dfrac{\partial \bar{f}}{\partial t}$:

Definition 2.1.1 (Exit set B of a circle-valued function) Consider $\dfrac{\partial f}{\partial t}$ as a smooth function on M, then B can be defined as follows:

$$B = \{x \in \partial M : -\frac{\partial \bar{f}}{\partial t}(x, 0) \leq 0\}.$$

2.2 The chain complex

Suppose $0 \in \mathbb{R}$ is a regular value of $\bar{f}$, let us set $N = \bar{f}^{-1}(\{0\}), M_N = \bar{f}^{-1}([0, 1])$ and $N' = \bar{f}^{-1}(\{1\})$. We can choose a generator $t \in G$ such that $t(N') = N$, in other words, $f_*(t) = -1$. Consider the negative half of the covering space $\bar{f}^{-1}((-\infty, 1])$ and denote it $\bar{M}^- = \bar{f}^{-1}((-\infty, 1])$. Let $M_j^1 = \bar{f}^{-1}([-j, 1])$ and $\bar{B}_j = \bar{B} \cap M_j^1$ for integer $j \geq 0$. Now to define the chain complex for circle-valued function f, we need the Smale transversality condition for the stable and unstable manifolds of the critical points. In this case, in the covering space, there are potentially infinitely many intersections to rectify, the Baire space argument in [35, Chapter 4] can be adapted to our situation and prove the existence of a Smale transverse vector field in the covering space.

Now we are ready to define the chain complex: As the critical points of f repeat themselves in each copy of M in the covering space $\bar{M}$, the modification of the filtration in Definition 1.3.9 is also effective here in describing the chain complex of M_j^1: Let

$$C_i(j) = \bar{M}^- - \bigcup_{k=0}^{j} \bigcup_{\text{ind}\,(p)\geq i+1} W^u(t^k p, v),$$

and then we define

$$C_i^{\text{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}) = H_i(C_i(j), C_{i-1}(j)).$$

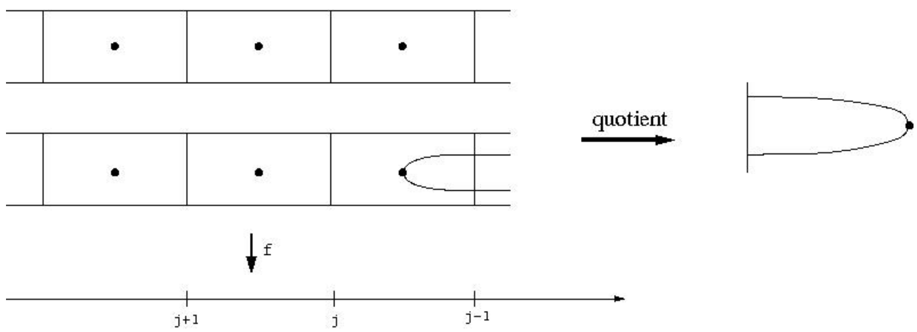

Figure 2.1: $C_*^{\text{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}) = \bigoplus_{k=1}^{j} \mathbb{Z}(t^k p)$

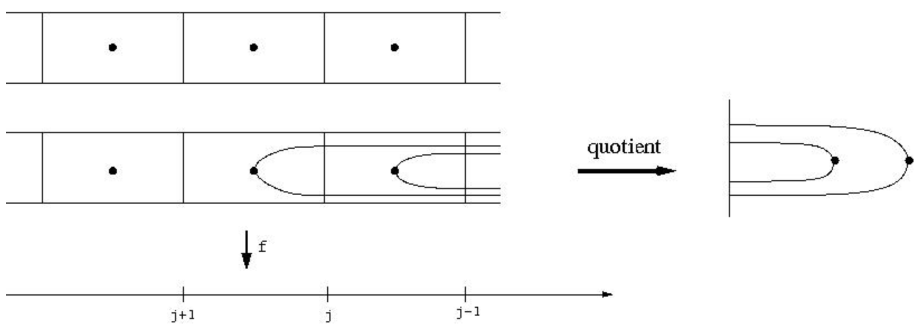

Figure 2.2: $C_*^{\text{MS}}(M_{j+1}^1, \bar{B}_{j+1} \cup M_{j+1}, \bar{f}, \bar{v}) = \bigoplus_{k=1}^{j+1} \mathbb{Z}(t^k p)$

Since the negative gradient flow of $-\bar{v}$ travels in the negative direction, and it intersects $M_j = \bar{f}^-(\{j\})$ transversely. So for each j, by Lemma 1.2.4 we can also carry over the isomorphism statement of Chapter 1 to here:

$$C_*^{\Delta}(M_j^1, \bar{B}_j \cup M_j) \simeq C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}).$$

Here each C_i^{MS} is generated by the critical points of $\bar{f}$ in M_j^1. Moreover, it can be seen as a $\mathbb{Z}[t]/t^j$ module generated by critical points of f, summarised in the following formula:

$$C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}) = \bigoplus_{\bar{p} \in \mathrm{Crit}_i(\bar{f})} \mathbb{Z}(\bar{p}) = \bigoplus_{k=1}^{j} \bigoplus_{p \in \mathrm{Crit}_i(f)} \mathbb{Z}(t^k p),$$

where $\mathbb{Z}(p)$ is a copy of $\mathbb{Z}$ generated by the critical point p.

Compare Figure 2.1 and Figure 2.2 where we increase the value j to $j+1$, then the chain complex becomes a $\mathbb{Z}[t]/t^{j+1}$ module instead of $\mathbb{Z}/t^j$.

And for each j, the boundary map $\partial_i^{(j)} : C_i^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}) \to C_{i-1}^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v})$ is

$$\partial_i^{(j)}(\bar{p}) = \sum_{\bar{q}, \mathrm{ind}\, \bar{q} = i} [\bar{p} : \bar{q}]\bar{q}.$$

It is well-defined as in Chapter 1.

Viewing such Morse chain complex $C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v})$ as a $\mathbb{Z}[t]/t^j$ module, we can read the boundary map over $\mathbb{Z}[t]/t^j$, namely, with uniquely chosen lifts $\bar{p}, \bar{q}$ of p, q:

$$\partial_i^{(j)}(\bar{p}) = \sum_{q \in \mathrm{Crit}_{i-1} f} \sum_{k=0}^{j-1} [\bar{p} : t^k \bar{q}] t^k \bar{q},$$

where $[\bar{p} : t^k \bar{q}]$ is the number of trajectories from p to $t^k \bar{q}$ according to sign.

Now we have a natural inverse system:

$$\phi_{j,i} : C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}) \to C_*^{\mathrm{MS}}(M_i^1, \bar{B}_i \cup M_i),$$

where $\phi_{j,i}$ is the projection from a $\mathbb{Z}[t]/t^j$ module to a $\mathbb{Z}[t]/t^i$ module, for $j > i \geq 0$.

Suppose we have only one critical point in $M_i - \mathrm{Int}\, M_{i-1}$, taking the inverse limit of such inverse system, we get

$$\begin{aligned}
\varprojlim C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}) &= \varprojlim \mathbb{Z}[t]/t^j \\
&= \left\{ (x_1, \dots, x_j, \dots) \in \prod_{j=1}^{\infty} \mathbb{Z}[t]/t^{j+1} : x_j = \phi_{j+1,j}(x_{j+1}) \right\} \\
&\overset{\Phi}{\cong} \mathbb{Z}[[t]].
\end{aligned}$$

The last isomorphism comes from map the $\Phi : \mathbb{Z}[[t]] \to \varprojlim \mathbb{Z}[t]/t^j$ as

$$\Phi(x) = \Phi(\sum_{j=0}^{\infty} a_j t^j) = (a_0, a_0 + a_1 t, \dots, \sum_{j=0}^{n} a_j t^j, \dots).$$

This shows the inverse limit is a $\mathbb{Z}[[t]]$ module, and the argument for several critical points work similarly. Finally we need another ring to tensor with this module to finish the preparation:

Definition 2.2.1 We define the *Novikov ring* as follows:

$$\mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}] = \{\lambda = \sum_{-\infty}^{\infty} a_j t^j : |\{t^j \text{ with } a_j \neq 0, j \leq 0\}| < \infty\},$$

where $|\{\ \}|$ is the cardinal number of a set.

Now we define the relative Novikov complex of a circle-valued function as follows:

Definition 2.2.2 We define the *relative Novikov complex* of a circle-valued function f as follows:

$$C_*^{\mathrm{Nov}}(\bar{M}, \bar{B}, f, v) = \mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} \varprojlim C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v})$$

The *boundary map* $\partial_i : C_i^{\mathrm{MS}}(\bar{M}, \bar{B}, f, v) \to C_{i-1}^{\mathrm{MS}}(\bar{M}, \bar{B}, f, v)$ is expressed as:

$$\partial_i(p) = \sum_{q \in \mathrm{Crit}_{\,i-1}(f)} \sum_{j=-\infty}^{\infty} [\bar{p} : t^j \bar{q}] t^j q,$$

where p, q are critical points of f in M with index $\mathrm{ind}\,(p) = i$ and $\mathrm{ind}\,(q) = i - 1$, and $\bar{p}, \bar{q}$ are chosen lifts of p and q in the minimal covering $\bar{M}$.

Remark 2.2.3 If $\bar{f}(t^j \bar{q}) > \bar{f}(\bar{p})$ then $[\bar{p} : t^j \bar{q}] = 0$ as the trajectories of the negative gradient only travel in the negative direction, then for any choice of the lifting of the critical points p_i of f on M, there will only be finitely many $t^j \bar{q}$ with $j \leq 0$ such that $[\bar{p} : t^j \bar{q}]$ are not trivial, i.e. $\sum_{j=-\infty}^{\infty} [\bar{p} : t^j \bar{q}] t^j \in \mathbb{Z}((t))$, which confirms the boundary map is well-defined.

2.3 The chain homotopy equivalence

We want to prove the chain homotopy equivalence between the simplicial complex of the manifold and the Novikov complex of a circle-valued function.

Similar to the real function case, we firstly need a chain map from the simplicial chain complex of the manifold to the Novikov complex by some smooth triangulation *adjusted* to the gradient v. For each j, we already have such chain map $\varphi_v^j : C_*^\Delta(M_j^1, \bar{B}_j \cup M_j) \to C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v})$ by a well chosen triangulation adjusted to v. Recall in the first chapter, we show how to perturb a given triangulation so that it is adjusted to v, and it turns out this is a generic property. As in [38], we can always pick an adjusted triangulation Δ in the base manifold and lift it up to the covering, hence define the chain map $\varphi_v = \mathrm{id} \otimes \varprojlim \varphi_v^j : \mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} C_*^\Delta(\bar{M}, \bar{B}) \to C_*^{\mathrm{Nov}}(\bar{M}, \bar{B}, f, v)$ as:

$$\varphi_v(\sigma) = \sum_{p, \mathrm{ind}\, p = \dim \sigma} \sum_{j=-\infty}^{\infty} [\bar{\sigma} : t^j \bar{p}] t^j p.$$

Proposition 2.3.1 The chain map φ_v is a chain homotopy equivalence:

$$\mathbb{Z}((t)) \otimes_{\mathbb{Z}[t]} C_*^\Delta(\bar{M}, \bar{B}) \simeq C_*^{\mathrm{Nov}}(\bar{M}, \bar{B}, f, v).$$

Proof: According to the results in Chapter 1, we have the homotopy equivalence

$$C_*^\Delta(M_j^1, \bar{B}_j \cup M_j) \simeq C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v})$$

for any finite $j \geq 0$ or $C^{\mathrm{MS}}(j) \simeq C^\Delta(j)$ with shorthand $C^{\mathrm{MS}}(j) = C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v})$ and $C^\Delta(j) = C_*^\Delta(M_j^1, \bar{B}_j \cup M_j)$. Viewing $C^{\mathrm{MS}}(j)$ and $C^\Delta(j)$ as $\mathbb{Z}[t]/t^j$ modules, we also has the following commutative diagram for any two $j > i \geq 0$:

$$\begin{tikzcd} C^\Delta(j) \arrow[r] \arrow[d, "\varphi_v^j"] & C^\Delta(i) \arrow[d, "\varphi_v^i"] \\ C^{\mathrm{MS}}(j) \arrow[r, "\phi_{ji}"] & C^{\mathrm{MS}}(i) \end{tikzcd},$$

where $\phi_{j,i}$ is the projection from a module over $\mathbb{Z}[t]/t^j$ to a module over $\mathbb{Z}[t]/t^i$, and φ_v^j is the homotopy equivalence, which induces isomorphism between the homology groups $\varphi_*^j : H(C^\Delta(j)) \cong H(C^{\mathrm{MS}}(j))$. Hence $\varprojlim H(C^\Delta(j)) \cong \varprojlim H(C^{\mathrm{MS}}(j))$ and $\varprojlim^1 H(C^\Delta(j)) \cong \varprojlim^1 H(C^{\mathrm{MS}}(j))$. Also the commutativity of the diagram implies chain map $\varprojlim \varphi_v^j : \varprojlim C^{\Delta}(j) \to \varprojlim C^{\mathrm{MS}}(j)$. Now since $\phi_{ji} : C^{\mathrm{MS}}(j) \to C^{\mathrm{MS}}(i)$ is surjective, i.e. satisfying the Mittag-Leffler condition automatically, and Theorem 3.5.8 in [42] tells us the following exact sequences:

$$0 \to \varprojlim{}^1 H_{i+1}(C^{\mathrm{MS}}(j)) \to H_i(\varprojlim C^{\mathrm{MS}}(j)) \to \varprojlim H_i(C^{\mathrm{MS}}(j)) \to 0,$$

and similarly,

$$0 \to \varprojlim{}^1 H_{i+1}(C^{\Delta}(j)) \to H_i(\varprojlim C^{\Delta}(j)) \to \varprojlim H_i(C^{\Delta}(j)) \to 0.$$

Therefore applying the Five Lemma to the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim{}^1 H_{i+1}(C^{\Delta}(j)) \arrow[r] \arrow[d, "\cong"] & H_i(\varprojlim C^{\Delta}(j)) \arrow[r] \arrow[d] & \varprojlim H_i(C^{\Delta}(j)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \varprojlim{}^1 H_{i+1}(C^{\mathrm{MS}}(j)) \arrow[r] & H_i(\varprojlim C^{\mathrm{MS}}(j)) \arrow[r] & \varprojlim H_i(C^{\mathrm{MS}}(j)) \arrow[r] & 0.
\end{tikzcd}$$

we have shown the isomorphism in the middle term $H_i(\varprojlim C^{\Delta}(j)) \to H_i(\varprojlim C^{\mathrm{MS}}(j))$ for each i.

Now since C^{Δ} and C^{MS} are free chain complexes, isomorphism in homology H_i for each i yields the chain homotopy equivalence:

$$C_*^{\Delta}(\bar{M}, \bar{B}) \overset{\varprojlim \varphi_v^j}{\simeq} \varprojlim C_*^{\mathrm{MS}}(M_j^1, \bar{B}_j \cup M_j, \bar{f}, \bar{v}).$$

Therefore $\varphi_v = \mathrm{id} \otimes \varprojlim \varphi_v^j$ is a chain homotopy equivalence. $\square$

2.4 Universal covering

In this section, we want to consider a bigger covering space, namely the universal covering. Using the same inverse limit argument, we construct a relative chain complex from the universal covering, and similarly obtain the homotopy equivalence result.

Suppose $f_* : \pi_1(M) \to \mathbb{Z}$ derived from the circle-valued function $f : M \to S^1$ is surjective as in the beginning of the chapter, let us consider the universal covering $\rho : \tilde{M} \to M$ which factors through the minimal covering $\bar{\rho} : \bar{M} \to M$ as a composition of the covering projection $\tilde{\rho} : \tilde{M} \to \bar{M}$ and $\bar{\rho}$, $\rho = \bar{\rho}\tilde{\rho}$. Let $G = \pi_1(M)$ be as before and $H = \pi_1(\bar{M}) = \ker f_*$ denote the fundamental group of the minimal total space $\bar{M}$, then we have covering transformation groups $\mathbb{Z}, H$ and G for $\bar{\rho}, \tilde{\rho}$ and ρ, respectively. Since $H = \ker f_*$ is the kernel of the homomorphism $f_* : G \to \mathbb{Z}$, it is a normal subgroup of G, $H \lhd G$, which together with $\mathbb{Z}$ gives us a short exact sequence

$$1 \to H \to G \to \mathbb{Z} \to 1.$$

So we reconstruct G with subgroups H and $\mathbb{Z}$ as a semidirect product:

$$G = H \rtimes_{\varphi} \mathbb{Z},$$

where $\varphi : H \to H$ is an automorphism as $\varphi(h) = t^{-1}ht$ for $h \in H$ and $t \in \mathbb{Z}$ as a generator of $\mathbb{Z}$ with $\xi(t) = -1$.

Similar to the situation in the minimal covering for $\bar{f} : \bar{M} \to \mathbb{R}$, we get a lift of $f\rho : \tilde{M} \to S^1$ to $\tilde{f} : \tilde{M} \to \mathbb{R}$ for the universal covering $\tilde{M}$. In the covering space $\tilde{\rho} : \tilde{M} \to \bar{M}$, for lifts $\tilde{N}, \tilde{N}'$ of $N = \bar{f}^{-1}(\{0\}), N' = \bar{f}^{-1}(\{1\})$ we still have $t(\tilde{N}') = \tilde{N}$, therefore for each $j \geq 0$, lifts $\tilde{M}_j^1, \tilde{M}_j$ and $\tilde{B}_j$ of $M_j^1 = \bar{f}^{-1}((-j, 1]), M_j = \bar{f}^{-1}(\{-j\})$ and $\bar{B}_j = \bar{B} \cap \bar{f}^{-1}((-j, 1])$ induce Morse chain complex $C_*^{\mathrm{MS}}(\tilde{M}_j^1, \tilde{B}_j \cup \tilde{M}_j, \tilde{f}, \tilde{v})$ as a $\mathbb{Z}H_{\varphi}[t]/t^j$ module.

Definition 2.4.1 We define the *Novikov ring* as follows:

$$\mathbb{Z}H_{\varphi}((t)) = \mathbb{Z}H_{\varphi}[[t]][t^{-1}] = \{\lambda = \Sigma_{-\infty}^{\infty} a_j t^j : |\{t^j \text{ with } a_j \neq 0, j \leq 0\}| < \infty\},$$

where $a_j \in \mathbb{Z}H$.

Definition 2.4.2 We define the *relative Novikov complex* as follows:

$$C_*^{\mathrm{Nov}}(\tilde{M}, \tilde{B}, f, v) = \mathbb{Z}H_{\varphi}((t)) \otimes_{\mathbb{Z}H_{\varphi}[[t]]} \varprojlim C_*^{\mathrm{MS}}(\tilde{M}_j^1, \tilde{B}_j \cup \tilde{M}_j, \tilde{f}, \tilde{v})$$

We also define the boundary map $\partial_i : C_i^{\mathrm{Nov}}(\tilde{M}, \tilde{B}, f, v) \to C_{i-1}^{\mathrm{Nov}}(\tilde{M}, \tilde{B}, f, v)$ as

$$\partial_i(p) = \sum_{q \in \mathrm{Crit}_{i-1}(f)} \sum_{j=-\infty}^{\infty} \left([\tilde{p} : t^j \tilde{q}] t^j\right) q,$$

where $p \in \mathrm{Crit}_i(f)$ in M and $\tilde{p}, \tilde{q}$ are the chosen lifts of p, q in the universal covering $\tilde{M}$.

By similar argument as in the minimal covering situation, we obtain the chain homotopy equivalence for this chain complex:

Proposition 2.4.3 The relative Novikov complex of a manifold M with exit set B of a circle-valued function f is chain homotopic equivalent to the relative simplicial complex:

$$\varphi_v : C_*^{\mathrm{Nov}}(\tilde{M}, \tilde{B}, f, v) \simeq \mathbb{Z}H_\varphi((t)) \otimes_{\mathbb{Z}H_\varphi[[t]]} \varprojlim C_*^\Delta(\tilde{M}_j^1, \tilde{B}_j \cup \tilde{M}_j),$$

where φ_v is defined as a chain map by counting the intersection number of the simplices of $C_*^\Delta(\tilde{M}_j^1, \tilde{B}_j \cup \tilde{M}_j)$ with unstable manifolds of critical points of $\bar{f}$, as j tends to infinity:

$$\varphi_v(\sigma) = \sum_p \sum_{j=-\infty}^{\infty} \left([\tilde{\sigma} : t^j \tilde{p}]t^j\right) p.$$

with the outer sum ranging over critical points p of f having index ind $(p) = \dim(\sigma)$.

□

Chapter 3

Closed 1-forms

We discuss the Novikov complex of a general Morse closed 1-form in this chapter. After the rational approximation of a closed 1-form, the boundary map of the corresponding Novikov complex can be described as in the circle-valued case, and then we adapt a Latour trick to show the homotopy equivalence.

3.1 Connection with the Morse complex

In this chapter we consider a general closed 1-form on a manifold M with boundary ∂M. A 1-form ω is said to be closed if $\mathrm{d}\omega = 0$, i.e. ω is a 1-cocycle in the de Rham cochain complex. Also according to the Poincaré Lemma, locally ω can be represented by a real function up to some constant added. So if $\{U\}$ is an open cover of M where each component U is simply connected (e.g. open convex subsets of Euclidean space $\mathbb{R}^m$), then there exists $f_U : U \to \mathbb{R}$ such that $\omega|_U = \mathrm{d}f_U$ for each U. With this property in hands, a closed 1-form is said to be *Morse* if locally each $f_U : U \to \mathbb{R}$ with $\omega|_U = \mathrm{d}f_U$ is Morse, i.e. whose critical points are nondegenerate. Subsequently, ω inherits the notion of *index* and the property of having finitely many critical points in a compact manifold M with boundary ∂M.

Also a closed 1-form ω induces a cohomology class $\xi : \pi_1(M) \to \mathbb{R}$ via homomorphism $\xi([\gamma]) = \int_\gamma \omega$, with γ a smooth loop representing the homotopy class $[\gamma] \in \pi_1(M)$. Notice the integral is independent of the choice of loops and closed 1-forms within the cohomology class, according to Stoke's theorem. Since the compactness of M induces that $\pi_1(M)$ is finitely presented, the image $\operatorname{Im}\xi$ of ξ is a finitely generated free abelian subgroup of $\mathbb{R}$, so $\operatorname{Im}\xi = \mathbb{Z}^k$ for some $k \in \mathbb{Z}^+$. Then if $k = 0$, ξ is trivial and ω is an exact form induced from some real function $f : M \to \mathbb{R}$ and we are in the case of Chapter 1.

When $k = 1$ and $\operatorname{Im}\xi = \mathbb{Z}$, we call ω *rational*. Let $\bar{\rho} : \bar{M} \to M$ be a regular covering space of M, such that $\pi_1(\bar{M}) = \ker\xi$. Then the covering transformation group is $\mathbb{Z}$ and $\xi = [\omega]$ is trivial on $\bar{\rho}_*(\pi_1(\bar{M}))$. Therefore $\bar{\rho}^*\omega = \mathrm{d}\bar{f}$ for some $\bar{f} : \bar{M} \to \mathbb{R}$. Now suppose $t \in \mathbb{Z}$ is a generator of the covering transformation group, such that $c = \bar{f}(\bar{x}) - \bar{f}(t\bar{x}) > 0$ for all $\bar{x} \in \bar{M}$, then we can establish the equivalence between a rational closed 1-form and a circle-valued function by defining $f : M \to S^1$ by $f(x) = \exp\big(2\pi i \bar{f}(\bar{x})/c\big)$, where $\bar{x} \in \bar{M}$ is a lift of $x \in M$, as we have $\omega = cf^*(\mathrm{d}\theta)$ where $\mathrm{d}\theta$ is the canonical angular form. So the case when $k = 1$ we refer to the preceding chapter on circle-valued functions.

Now assume $k > 1$, in which case we call the closed 1-form ω *irrational*. We want to establish the homotopy equivalence on the relative chain complexes analogous to the previous ones.

Similar to the circle-valued function, the exit set of a closed 1-form ω is defined by the partial derivative of the pullback f of ω:

Definition 3.1.1 (Exit set of a closed 1-form ω) Let $\rho : \bar{M} \to M$ be a regular covering space of M corresponding to the kernel $\ker[\omega]$ of ω, then *the exit set of* ω is defined as:

$$B = \{x \in \partial M : \frac{\partial f}{\partial t}|_{(x,0)} \leq 0\},$$

where $f : \bar{M} \to \mathbb{R}$ is a real function such that $\mathrm{d}f = \rho^*\omega$.

3.2 Novikov ring

Consider the universal covering space $\tilde{\rho} : \tilde{M} \to M$, such that $\tilde{\rho}^*([\omega]) = 0$. For the construction of a general Novikov complex, just as what happens in the circle-valued function case, we start with some more sophisticated coefficient ring, then get the appropriate boundary map:

Definition 3.2.1 (Novikov ring $\widehat{\mathbb{Z}G}_\xi$) Given a group G and a homomorphism $\xi : G \to \mathbb{R}$, then we define the *Novikov ring* as:

$$\widehat{\mathbb{Z}G}_\xi = \{\lambda = \Sigma\, a_g g \in \mathbb{Z}^G : |\{a_g \neq 0, \xi(g) > K\}| < \infty, \text{ for all } K \in \mathbb{R}\},$$

where $a_g \in \mathbb{Z}$, $g \in G$ are generators of G.

Remark 3.2.2 Notice that when the map $\xi : G \to \mathbb{Z}$ is surjective, the above definition coincides with the Novikov ring $\mathbb{Z}H_\varphi((t))$ defined in the previous chapter. We make this precise here:

Choose an element $t \in G$ of the additive group $\mathbb{Z}$ such that $\xi(t) = -1$, and write $H = \ker \xi \lhd G$, then we identify G as a semidirect product of H and $\mathbb{Z} = \langle t \rangle$ as $G = H \rtimes_\varphi \mathbb{Z}$ where $\varphi : H \to H$ is an automorphism $\varphi(h) = t^{-1}ht$, so that $ht = t\varphi(h)$. Then for each element $h \in H$ and $n \in \mathbb{Z}$, there exists unique $g \in G$, such that g can be written as $g = ht^n$. Then

$$\xi(g) = \xi(ht^n) = \xi(h)\xi(t^n) = -n$$

implies that for each element $\lambda = \sum_g a_g g \in \mathbb{Z}^G$, the number of nontrivial a_g with $\xi(g) > K$ for any $K \in \mathbb{R}$ is finite, if the number of corresponding nontrivial $a_{n,h}$ with $n < 0$ is finite in $\sum_{n,h} a_{n,h} ht^n$ with $ht^n = g$. In other words, the map sending ht^n to $g = ht^n \in G$ induces an isomorphism between $\mathbb{Z}H_\varphi((t))$ and $\mathbb{Z}G_\xi$.

3.3 Novikov complex

We now turn to the definition of Novikov complex that generalises the idea of Morse complex to the case of closed 1-forms.

Consider the universal covering $\tilde{\rho} : \tilde{M} \to M$, choose a real Morse function $\tilde{f} : \tilde{M} \to \mathbb{R}$ such that $\tilde{\rho}^*(\omega) = \mathrm{d}\tilde{f}$ and a lift $\tilde{p}$ in $\tilde{M}$ for each critical point $p \in M$. Then the chain complex is potentially generated by all the critical points in the covering, together with the Novikov ring defined in the previous chapter, we want to show the boundary is well-defined. We do this by a rational approximation lemma. Because $\mathbb{Q} \subset \mathbb{R}$ is dense in $\mathbb{R}$, the idea is to slightly perturb the original closed 1-form without changing the gradient vector field with respect to some Riemannian metric, so that the map $\xi : G = \pi_1(M) \to \mathbb{R}$ is factorised through $\mathbb{Q}$. Now since any $\xi \in \mathrm{Im}\,(G \to \mathbb{Q})$ can only have rank 1, this leads us back to the circle-valued case, where the boundary is already defined. The approach is based on [38].

Lemma 3.3.1 Suppose we have compact manifold M with boundary ∂M. Let ω be a Morse closed 1-form on M with exit set $B \subset \partial M$ and v be a gradient vector field of ω with respect to some Riemannian metric, then there exists a rational closed 1-form ω' with the same gradient vector field v for a chosen Riemannian metric, such that it coincides with ω in some small neighbourhoods of the critical points of ω. Moreover, the cohomology class of ω', denoted as ξ' vanishes in the kernel $\ker \xi$ of $\xi = [\omega]$, the cohomology class of ω.

Proof: First observe that we only need to fix the rank problem for the whole manifold, as for the exit set B, $i^*\xi : \pi_1(B) \to \mathbb{R}$ filters through $\xi : \pi_1(M) \to \mathbb{R}$, so if ω is rational for the whole manifold M, it will be rational in B automatically.

We choose the minimal set of generators of $\mathrm{Im}\,\xi \subset \mathbb{R}$, namely, $g_1, \ldots, g_k \in \mathrm{Im}\,\xi$. Define Morse closed 1-forms $\omega_1, \ldots, \omega_k$ such that for each $i = 1, \ldots, k$ the cohomology class $\xi_i = [\omega_i] : G \to \mathbb{R}$ satisfies $\xi_i(g_j) = 1$ when $i = j$ and $\xi_i(g_j) = 0$ otherwise and ξ_i vanishes on $\ker \xi$. And because the critical points of ω are isolated, we can also assume that ω_i vanishes in a small neighbourhood U_p of each critical point p of ω. For this is possible by first representing ω_i by some function f_p on the neighbourhood U_p and then extending it trivially to the whole manifold as F_p such that when we restrict F_p on a smaller neighbourhood V_p, it coincides with f_p, $F_p|_{V_p} = f_p|_{V_p}$. Finally subtracting F_p from ω_i to get $\omega_i - F_p$ and repeating such modification finitely many times will return us the desired situation.

Now choose $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \mathbb{R}^k$ with $\|\epsilon\|$ very small and define

$$\omega_\epsilon = \omega + \sum_{i=1}^{k} \epsilon_i \omega_i.$$

Then ω still dominates the expression, meaning:

$$\omega_\epsilon(v(x)) > 0, \tag{3.1}$$

for x with $\omega(x) \neq 0$. And for any vector field X on M, in a neighbourhood U_p of a critical point p,

$$\omega_\epsilon(X_x) = \langle X_x, v(x) \rangle,$$

for each $x \in U_p$, where $\langle\, ,\, \rangle$ is the original Riemannian metric. But we want to have v as a gradient vector field for ω_ϵ as well, so before moving on to the next stage, we need to modify the old Riemannian metric here. We again work on it locally by choosing a set $\{U_i\}$ of open subsets of M, so that together with the neighbourhoods of critical points, they form an open cover $\mathcal{U}$ of M. Now we have real function $f : U \to \mathbb{R}$ on each $U \in \mathcal{U}$ with $\mathrm{d}f = \omega_\epsilon|_U$. Then because ω is nonzero on U, in particular 0 is not a critical point of f, we can adjust f with a suitable local coordinate system near 0 so that f is a projection to the first coordinate, i.e. $f(x_1, \cdots, x_m) = x_1$.

Let us write $v = \sum_i v_i \dfrac{\partial}{\partial x_i}$ for $i = 1, \ldots, m$ and consider $v, \dfrac{\partial}{\partial x_2}, \ldots, \dfrac{\partial}{\partial x_m}$ as a basis for the vector fields on U. Then we can choose a Riemannian metric g so that this basis is orthogonal with respect to g, in particular, we define $g(v, v) = v_1 > 0$, which is true according to (3.1). With such Riemannian metric, it is easy to check that $u(f) = g(v, u)$ for any vector field u on U, i.e. v is a gradient of f and hence ω_ϵ restricted to U. Glue these local Riemannian metrics together, we obtain a Riemannian metric on M such that v is the gradient vector field of ω_ϵ.

So far, we haven't said anything of ω_ϵ on the boundary when picking all the ω_i, and now we take care of this issue. Namely, we want to adapt ω_i so they satisfy assumptions **A1**, **A2** and **A3** on B as well as ω. In particular, for a neighbourhood V_x of each point $x \in \Gamma$, function $\dfrac{\partial f}{\partial t}$ is regular on $(x, 0, 0)$, where $f_i : V_x \to \mathbb{R}$ is the exact form of ω_i on V_x.

Since $\omega_i|_{\Gamma\times[-1,1]\times[0,1]}$ is cohomologous to some ω_i' on Γ, i.e. $\omega_i|_\Gamma = \omega_i' + \mathrm{d}g$ for some $g : \Gamma \to \mathbb{R}$, extend g to the whole manifold M, name it g too, then $\omega_i'' = \omega_i - \mathrm{d}g$ is independent of the last coordinate t on Γ, therefore

$$\frac{\partial f_\epsilon}{\partial t} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f_i''}{\partial t} = \frac{\partial f}{\partial t},$$

and so it is regular on $(x, 0, 0)$ as $\dfrac{\partial f}{\partial t}$, where f_ϵ and f are local exact form of ω_ϵ and ω, respectively.

Hence, such ω_ϵ shares the same gradient vector field v with ω when ϵ_i are small, and have respect to the boundary conditions on B. Finally, for each g_i,

$$\omega_\epsilon(g_i) = \omega(g_i) + \epsilon_i \omega_i(g_i) \in \mathbb{Q}$$

when $\epsilon_i \in \mathbb{R}$ is carefully chosen for each i.

Set $\omega' = \omega_\epsilon$ and $\xi' = [\omega_\epsilon]$, $\xi' : G \to \mathbb{Q}$.

We have proved the lemma. $\square$

Definition 3.3.2 If a closed 1-form ω' is modified from a general Morse closed 1-form ω as the lemma above, we call ω' the *rational approximation* of ω.

Taking on this new closed 1-form and its cohomology class, we would have got the relative Novikov ring $C_*^{\mathrm{Nov}}(M, B, \omega', v)$ with local coefficients $\mathbb{Z}_\varphi((t))$, where $\tilde{\omega}$ is seen as a circle-valued function. However, in order to stick to the original closed 1-form and its Novikov ring, we only want to apply the approximation lemma to the boundary map, so that it is well-defined. In other words, we want to show that the boundary map $\partial : C_i(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v}) \to C_{i-1}(\tilde{M}, \tilde{B}, \tilde{f}, \tilde{v})$ as

$$\partial p \quad = \sum_{q \in \mathrm{Crit}\,_{i-1}(\omega)} [\tilde{p} : \tilde{q}] q, \tag{3.2}$$

where $\tilde{p}, \tilde{p}$ are critical points of $\tilde{f}$ of index $i-1$ whose projections coincide with $p, q \in \mathrm{Crit}\,(\omega)$, respectively: $\tilde{\rho}(\tilde{p}) = p, \tilde{\rho}(\tilde{q}) = q$, so that we have $[\tilde{p} : \tilde{q}] \in \widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$.

Lemma 3.3.3 Let ω and ω' be two Morse closed 1-forms that coincide in a neighbourhood of each critical point, and there is vector field v as the gradient of both 1-forms. Denote the cohomology classes $\xi, \xi' : \pi_1(M) \to \mathbb{R}$ of ω and ω' respectively, then there exists real constants A, B and C, D with $A > 0$ and $C > 0$ such that whenever there is $g \in G = \pi_1(M)$ and a trajectory $\tilde{\gamma} : \mathbb{R} \to \tilde{M}$ of $-\tilde{v}$ between $g\tilde{q}$ and $\tilde{p}$ with critical points $\tilde{p}, \tilde{q} \in \tilde{M}$, then

$$\xi(g) \leq A\xi'(g) + B$$

and

$$\xi'(g) \leq C\xi(g) + D.$$

Proof: Choose a path γ_{pq} from p to q in the base space M, such that when it is lifted up to the covering $\tilde{M}$, say $\gamma_{\tilde{p}\tilde{q}}$, together with γ from $\tilde{p}$ to $g\tilde{q}$, the loop $\gamma \cdot \gamma_{\tilde{p}\tilde{q}}^{-1} = \gamma \cdot \gamma_{\tilde{q}\tilde{p}}$ represents the homotopy class $g \in \pi_1(M)$ in the base space. See Figure 3.1 below

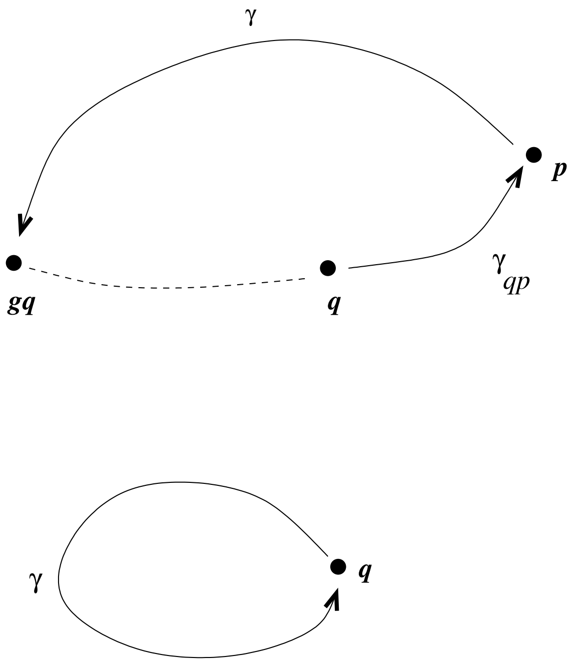

Figure 3.1: The lift of loop γ in the covering

According to the observation in the next lemma, there exists $K > 0$ such that $|\int_{\gamma_{pq}} \omega| \leq K$ and so does ω'. Moreover, since ω and ω' coincide near the critical points, by compactness there exists an L with $0 < L < 1$ such that $\omega(v(x)) \geq L\omega'(v(x))$ for all $x \in M$. Now consider:

$$\begin{aligned}
\xi(g) &= \int_{\gamma_{\tilde{q}\tilde{p}}} \omega + \int_{\gamma} \omega \\
&\leq K - \int_{-\infty}^{\infty} \omega(v(\gamma(t)))\mathrm{d}t \\
&\leq K - L\int_{-\infty}^{\infty} \omega'(v(\gamma(t)))\mathrm{d}t \\
&= K - L(\int_{\gamma_{\tilde{q}\tilde{p}}} \omega' - \int_{\gamma} \omega' - \int_{\gamma_{\tilde{q}\tilde{p}}} \omega') \\
&\leq K + LK + L\big(\int_{\gamma_{\tilde{q}\tilde{p}}} \omega' + \int_{\gamma} \omega'\big) \\
&= K(1+L) + L\xi'(g).
\end{aligned}$$

Setting $A = L$ and $B = K(1 + L)$ gives the first inequality; switch ω and ω' in the same argument we obtain the second inequality. $\square$

Corollary 3.3.4 Suppose ω is a Morse closed 1-form and ω' is the rational approximation of ω so that v is the transverse gradient vector field of both forms. Then $[\tilde{p} : \tilde{q}] \in \widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$ for $p, q \in \mathrm{Crit}\,(\omega)$ with $\mathrm{ind}\,(p) = \mathrm{ind}\,(q) + 1$. $\square$

Denote $C_*(M, B, \omega, v)$ to be a module generated by the critical points of ω, and when we consider it in the universal covering, it is naturally endowed with a $\mathbb{Z}G$ action. Since $\mathbb{Z}G$ sits inside $\widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$, we have a natural ring inclusion $\mathbb{Z}G \hookrightarrow \widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$ and $\widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$ can be seen as a right module of $\mathbb{Z}G$. So we can view this potential candidate $C_*(M, B, \omega, v)$ as a $\widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$ module and the boundary map can be defined within the intersection $\widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$ according to Corollary 3.3.4 above. Now tensoring with $\widehat{\mathbb{Z}G}_\xi$, notice that

$$\widehat{\mathbb{Z}G}_\xi \otimes_{\widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}} C_*(M, B, \omega, v) \simeq \widehat{\mathbb{Z}G}_\xi \otimes_{\widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}} C_*(M, B, \omega', v),$$

So by this way, we retain the original ring coefficients while define the boundary map according to the circle-valued model.

Definition 3.3.5 We define the *relative Novikov complex* as a $\widehat{\mathbb{Z}G}_\xi$-module generated by the critical points of ω:

$$C_*^{\mathrm{Nov}}(M, B, \omega, v) = \bigoplus_{p \in \mathrm{Crit}\,(\omega)} \widehat{\mathbb{Z}G}_\xi,$$

where the boundary map is defined as (3.2).

The rest of the chapter, we want to show the chain homotopy equivalence of the following map:

$$\varphi_v : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^{\Delta}(\tilde{M}, \tilde{B}) \to C_*^{\mathrm{Nov}}(M, B, \omega, v).$$

Here φ_v is defined similarly to the circle-valued case, and to end this section, we claim φ_v is a chain map by stating the coefficients of $\varphi_v(\sigma)$ lie in the intersection $\widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$:

Proposition 3.3.6 Choose a triangulation Δ adjusted to the gradient vector field v of ω, then the map $\varphi_v : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}, \tilde{B}) \to C_*^{\mathrm{Nov}}(M, B, \omega, v)$ is defined as

$$\varphi_v(\sigma) = \sum_{p \in \mathrm{Crit}_i(\omega)} [\tilde{\sigma} : \tilde{p}]p,$$

where $\sigma \in C_*^\Delta(M, B)$ with $\dim(\sigma) = i$ and $\tilde{\sigma}, \tilde{p}$ are translates of σ and p as $\tilde{\rho}(\tilde{\sigma}) = \sigma$ and $\tilde{\rho}(\tilde{p}) = p$. Then the coefficients $[\tilde{\sigma} : \tilde{p}] \in \widehat{\mathbb{Z}G}_\xi \cap \widehat{\mathbb{Z}G}_{\xi'}$. □

The proof goes similarly to Lemma 3.3.5 and we refer to [38, proposition 4.7] for details.

3.4 Continuation

Before we move on to show the homotopy equivalence of the *Novikov complex*, we need a relative version of the continuation argument.

Proposition 3.4.1 Let M be a compact manifold with boundary ∂M, and ω_i are cohomologous Morse closed 1-forms on M for $i = 0, 1$ such that ω_1, ω_2 lie in the same cohomology class ξ, $\xi = [\omega_i] \in H^1(M)$. Assume the exit sets of ω_0 and ω_1 coincide, denote $B = B_{\omega_0} = B_{\omega_1}$, then

$$C_*^{\mathrm{Nov}}(M, B, \omega_0) \simeq C_*^{\mathrm{Nov}}(M, B, \omega_1).$$

Notation 3.4.2 Here we suppress the vector fields of ω_i in the above statement. To further simplify the notations in the proof, we use $C_*^{\mathrm{Nov}(i)} = C_*^{\mathrm{Nov}}(M, B, \omega_i)$ and $M_i = M \times \{i\}$ for $i = 0, 1$.

The proposition says the chain homotopy type of the *Novikov complex* depends only on the cohomology class of the closed 1-form, its exit set and the covering space. The argument for closed manifolds is given in [36].

Proof: Since ω_0, ω_1 are in the same cohomology class, they only differ by an exact form, so let $g : M \to \mathbb{R}$ be the Morse function with the same exit set $B \subset \partial M$, such that $\omega_1 = \omega_0 + \mathrm{d}g$, and let $\tau : [-1, 1] \to [0, 1]$ be a real smooth function with $\tau(t) = 1$ for $t \in [-1, -1 + \delta] \cup [1 - \delta, 1]$ and $\tau(t) = 0$ for $t \in [-\delta', \delta']$, where $\delta, \delta' > 0$

are small. Then we get a homotopy from ω_0 to ω_1:

$$\omega_t = \omega_0 + \mathrm{d}(\tau(t)g).$$

Notice we have the same Novikov ring $\widehat{\mathbb{Z}G_\xi}$ where G is the fundamental group of M, according to the hypothesis. Moreover, when lifted up to the universal covering $\rho : \tilde{M} \to M$, the pullback $\rho^*(\omega_t)$ of ω_t is exact for each t, therefore, they correspond to some real functions f_t, or explicitly, there exists real smooth function $f_0 : \tilde{M} \to \mathbb{R}$ such that $\rho^*\omega_t = \mathrm{d}f_t = \mathrm{d}f_0 + \mathrm{d}\big(\tau(t)(g \circ \rho)\big)$.

Now on the total space of the covering, we build a real smooth function F upon f_t:

$$F : \tilde{M} \times S^1 \to \mathbb{R},$$

so for $(x,t) \in \tilde{M} \times S^1$, where $-1 \leq t \leq 1$ and S^1 is parametrized by $t \to e^{\pi i t}$,

$$F(x,t) = f(x,t) - \frac{K}{\pi}\cos \pi t.$$

Observe that the critical points of F are simply the ones of f_0 and f_1, with the indices of $\operatorname{Crit} f_1$ shifted one degree up as:

$$\frac{\partial^2 \cos \pi t}{\partial t^2} = K\pi \cos \pi t = -K\pi < 0, \text{ when } t = 1.$$

That is, for $p \in \operatorname{Crit} F \cap M_0$, then $\operatorname{ind}_F(p) = \operatorname{ind}_{f_0}(p)$ and for $p \in \operatorname{Crit} F \cap M_1$, $\operatorname{ind}_F(p) = \operatorname{ind}_{f_1}(p) + 1$.

Bringing the exact form dF down to the base space $M \times S^1$, it corresponds to a closed 1-form Ω. We restrict Ω to a closed subset $M \times [-\epsilon, 1+\epsilon] \subset M \times S^1$. Now let v_i be the gradients of ω_i with respect to chosen Riemannian metrics on M_i for $i = 0, 1$; and we extend v_i to $(v_i, K \sin \pi t \partial_t)$ on $M_i \times [-\delta_i + i, i + \delta_i]$. Then after choosing a Riemannian metric for the rest of $M \times [-\epsilon, 1+\epsilon]$, we glue them together using partition of unity. With respect to this global Riemannian metric, we have gradient vector field V of Ω on $M \times S^1$. Notice $V = (v_i, K \sin \pi t \partial_t)$ near M_i for $i = 0, 1$. We assume such vector field V is transverse.

Moreover, when considering the boundary

$$\partial(M \times [-\epsilon, 1+\epsilon]) = \partial M \times [-\epsilon, 1+\epsilon] \bigcup \big(M \times \{-\epsilon\} \cup M \times \{1+\epsilon\}\big),$$

we have

$$-\frac{\partial F}{\partial s}(x,0,t) = -\frac{\partial f_0}{\partial s} - \frac{\tau(t)\partial(g\circ\rho(x,s))}{\partial s} \leq 0 \Longleftrightarrow x \in \tilde{B},$$

for $(x,0,t) \in \partial M \times [0,1) \times [-\epsilon, 1+\epsilon]$, and

$$-\frac{\partial f}{\partial t}(x,-\epsilon) = K\sin(-\pi\epsilon) < 0,$$

for $(x,-\epsilon) \in \tilde{M} \times \{-\epsilon\}$. Therefore, we have $M \times \{-\epsilon\} \cup B \times [-\epsilon, 1+\epsilon]$ as the exit set of U and it forms a corner.

Consider the pair $(\tilde{M} \times [-\epsilon, 1+\epsilon], \tilde{M} \times \{-\epsilon\} \cup \tilde{B} \times [-\epsilon, 1+\epsilon])$ in the covering space, for each finite copies of M, it is a compact manifold with corners depicted in Figure 3.2:

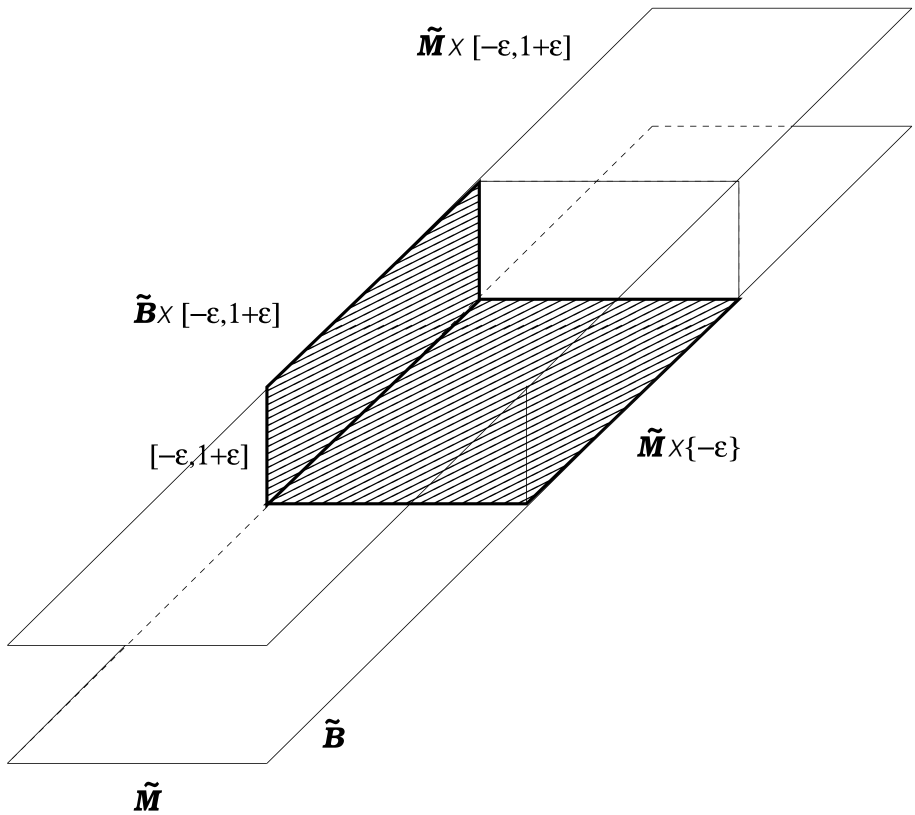

Figure 3.2: Manifold with corners (the shaded area)

Lemma 1.2.4 and Theorem 1.2.9 in Chapter 1 imply that each compact piece of $\tilde{M} \times S^1$ is homotopic to a CW complex generated by the critical points of F on that piece. Now apply the inverse limit techniques and Novikov ring developed in Chapter 2, we have chain complex of $\tilde{M} \times [-\epsilon, 1+\epsilon]$ as a $\mathbb{Z}G_\xi$ module generated by the critical points of Ω. Denote it as

$$C_*(M \times [-\epsilon, 1+\epsilon], M \times \{-\epsilon\} \cup B \times [-\epsilon, 1+\epsilon], \Omega).$$

In particular, since critical points of Ω exclusively come from ω_i for $i = 0, 1$, we can identify the chain complex of Ω on $M \times [-\epsilon, 1+\epsilon]$ by the following equalities:

$$\begin{aligned} & C_*(M \times [-\epsilon, 1+\epsilon], M \times \{-\epsilon\} \cup B \times [-\epsilon, 1+\epsilon], \Omega) \\ = & \bigoplus_{i=0}^{1} C_*(M \times [-\epsilon, 1+\epsilon], M \times \{-\epsilon\} \cup B \times [-\epsilon, 1+\epsilon], \omega_i) \\ \simeq & C_*^{\mathrm{Nov}(0)} \oplus C_{*-1}^{\mathrm{Nov}(1)} \end{aligned}$$

Accordingly, the boundary map ∂ can be split into the boundary maps ∂_0, ∂_1 of the two components $C_*^{\mathrm{Nov}}(M, B, \omega_0) = C^{\mathrm{Nov}(0)}, C_*^{\mathrm{Nov}}(M, B, \omega_1) = C^{\mathrm{Nov}(1)}$, respectively, and a map ϕ_{01} from $C^{\mathrm{Nov}(1)}$ to $C^{\mathrm{Nov}(0)}$ by counting the trajectories going from the critical points of M_1 to M_0. By the next Lemma 3.4.3, when K and δ are chosen carefully, we can guarantee there exist no trajectories from M_0 back to M_1, furthermore, ∂_i is precisely the boundary map of the chain complex of M_i for $i = 0, 1$. We summarise the information in the following matrix form:

$$\partial = \begin{pmatrix} \partial_0 & \phi_{01} \\ 0 & \partial_1 \end{pmatrix}$$

The 0 in the bottom left corner reflects the fact that there exist no trajectories from M_0 to M_1.

Now as boundary map, $\partial^2 = 0$ induces the equality $\partial_0\phi_{01} + \phi_{01}\partial_1 = 0$, so if we modify ϕ_{01} as the following:

$$\varphi_{01} = (-1)^k \phi_{01} : C_k^{\mathrm{Nov}(1)} \to C_k^{\mathrm{Nov}(0)}, \tag{3.3}$$

then φ_{01} becomes a chain map from $C_k^{\mathrm{Nov}(1)}$ to $C_k^{\mathrm{Nov}(0)}$.

Similarly, we can construct a Morse function on $\tilde{M} \times S^1 \times S^1$ based on the homotopy $\omega_{t,s} = \omega_0 + \mathrm{d}\tau(t)g + \mathrm{d}\kappa(s)h$ of the four cohomologous closed 1-forms ω_i for $i = 2\tau + \kappa$ when $\tau = 0, 1$ and $\kappa = 0, 1$. Here $\tau, \kappa : [-1, 1] \to [0, 1]$ are smooth functions with value 1 near -1 and 1, and value 0 near 0; and g, h are real functions on M with the same exit set B, so that $\omega_1 = \omega_0 + \mathrm{d}g$, $\omega_2 = \omega_0 + \mathrm{d}h$ and

$\omega_3 = \omega_0 + \mathrm{d}g + \mathrm{d}h$. Consider the closed subset $U = M \times [-\epsilon, 1+\epsilon]^2$, then the gradient flow of the pushforward Ω exits through $B \times [-\epsilon, 1+\epsilon]^2 \cup \bigcup_{i=0}^{1} M_i \times \{-\epsilon\} \times [-\epsilon, 1+\epsilon]$, so we obtain the decomposition of the chain complex

$$\begin{aligned} & C_*\Big(U, B \times [-\epsilon, 1+\epsilon]^2 \cup \bigcup_{i=0}^{1} M_i \times \{-\epsilon\} \times [-\epsilon, 1+\epsilon]\Big) \\ = \ & C_*^{\mathrm{Nov}(0)} \oplus C_{*-1}^{\mathrm{Nov}(1)} \oplus C_{*-1}^{\mathrm{Nov}(2)} \oplus C_{*-2}^{\mathrm{Nov}(3)} \end{aligned}$$

with boundary map as follows:

$$\partial = \begin{pmatrix} \partial_0 & \phi_{01} & \phi_{02} & \phi_{03} \\ 0 & \partial_1 & 0 & \phi_{13} \\ 0 & 0 & \partial_2 & \phi_{23} \\ 0 & 0 & 0 & \partial_3 \end{pmatrix}.$$

Now $\partial^2 = 0$ induces similar anticommutative property for ϕ_{ij}. It becomes a chain map after modification of sign, denoted φ_{ij}.

the following diagram summarises the information on chain maps:

$$\begin{tikzcd} C_*^{\mathrm{Nov}(1)} \arrow[d, "\varphi_{01}"'] & C_*^{\mathrm{Nov}(3)} \arrow[l, "\varphi_{13}"'] \arrow[dl, "\varphi_{03}"'] \arrow[d, "\varphi_{23}"] \\ C_*^{\mathrm{Nov}(0)} & C_*^{\mathrm{Nov}(2)} \arrow[l, "\varphi_{02}"'] \end{tikzcd} . \tag{3.4}$$

Moreover, we have the equation:

$$\varphi_{03} \circ \partial_0 + \varphi_{13} \circ \varphi_{01} + \varphi_{23} \circ \varphi_{02} + \partial_3 \circ \varphi_{03} = 0$$

this is essentially a homotopy equivalence between chain maps $\varphi_{13} \circ \varphi_{01}$ and $\varphi_{23} \circ \varphi_{02}$ under homotopy φ_{03}. Therefore if we identify $\omega_3 = \omega_2 = \omega_0$ in the diagram (3.4), we have shown that φ_{01} has a left inverse, repeat similar argument, we can find a right inverse so that φ_{03} is a chain homotopy equivalence. And this proves the statement.

□

Lemma 3.4.3 Following the notations of the theorem above, there exist K and δ so that the gradient flow travels from M_1 to M_0 one way, in particular, if $p \in M_0$ is a critical point in M_0, then the stable and unstable manifolds of p are contained in M_0: $W^s(p, V) \subset M_0$ and $W^u(p, V) \subset M_0$.

Proof: We only need to show it is true for the case of $i = 0$. Since $\tau : [-1, 1] \to [0, 1]$ can be chosen so that $\tau(t) = 0$ for any $t \in [-\delta, \delta]$, where δ is positive, then $\dfrac{\partial f(x,t)}{\partial t} = 0$ for any point lies within $M_0 \times [-\delta, \delta]$. Therefore,

$$\frac{\partial F}{\partial t}(x, t) = \frac{\partial f(x,t)}{\partial t} + K \sin \pi t = K \sin \pi t \geq 0,$$

and in particular, there is no flow going out of M_0. $\square$

3.5 Homotopy equivalence(with the Latour trick)

Now we are ready to show the homotopy equivalence between the Novikov complex and the usual chain complex of the pair (M, B)(e.g. simplicial complex). When applying a nice trick of Latour [22], extra attention is needed to make sure the supplementary exact form we are constructing have the exit set coincide with the original one. Let us do this in the following:

Lemma 3.5.1 Let ω be a closed 1-form one M with boundary ∂M and B be the exit set of ω, then there always exists a Morse function $F : M \to \mathbb{R}$ on M whose exit set B_F coincides with B_ω, $B_F = B_\omega$.

Proof: Choose a collaring for ∂M in $M^+ = M \cup_{\partial M} \partial M \times [1, 2)$, denote it $\partial M \times (0, 2)$ such that $\partial M \times \{1\} \cong \partial M$ and $(x, t) \in \mathrm{Int}\,(M)$ for $0 \leq t < 1$ and $(x, t) \in M^+ \setminus M$ for $1 < t \leq 2$. Let us cover $\partial M \times (0, 2)$ by $\{\mathrm{Int}\,(T) \times (0, 2), \mathrm{Int}\,(B) \times (0, 2), \partial B \times (-1, 1) \times (0, 2)\}$, where $\partial B \times (-1, 1)$ is an tubular neighbourhood of ∂B in ∂M, with $\partial B \times (-1, 0] \subset B$ and $\partial B \times [0, 1) \subset T$, where $T = \overline{\partial M - B}$.

Now define

$$g^+ : \mathrm{Int}\, T \times (0, 2) \to \mathbb{R} \text{ as } g^+(x, t) = -t$$

and

$$g^- : \mathrm{Int}\, B \times (0,2) \to \mathbb{R} \text{ as } g^-(x,t) = t$$

and

$$g : \partial B \times (-1,1) \times (0,2) \to \mathbb{R} \text{ as } g(x,s,t) = -t\sin(\pi s).$$

We want g to satisfy the boundary conditions **A1**, **A2** and **A3**, namely,

$$\begin{aligned} \frac{\partial g}{\partial t}|_{\partial B\times\{0\}\times\{1\}} &= 0, \\ \frac{\partial g}{\partial t}|_{\partial B\times(0,1)\times\{1\}} &< 0, \\ \frac{\partial g}{\partial t}|_{\partial B\times(-1,0)\times\{1\}} &> 0; \end{aligned}$$

and

$$\frac{\partial g}{\partial s} > 0;$$

and when we apply a partition of unity, the overall function $G = \phi_1 g^+ + \phi_2 g + \phi_3 g^-$ does not produce extra critical points on ∂M for suitable $\{\phi_i\}$ subordinate to $\{\mathrm{Int}\,(T) \times (0,2), \mathrm{Int}\,(B) \times (0,2), \partial B \times (-1,1) \times (0,2)\}$, where $0 \le \phi_i \le 1$ for $i = 1,2,3$.

Check:

$$\frac{\partial g}{\partial t} = \sin(\pi s) = 0 \text{ iff } s = 0,$$

and

$$-\frac{\partial g}{\partial t} = -\sin(\pi s) \begin{cases} < 0 & 0 < s \le 1 \\ > 0 & -1 \le s < 0 \end{cases},$$

and

$$\frac{\partial g}{\partial s} = \pi\cos(\pi s) = \pi\cos(\pi s) = \pi\cos 0 = \pi > 0.$$

Now set

$$G = \phi_1 g^+ + \phi_2 g + \phi_3 g^-.$$

Notice there are four overlaps in the cover, as illustrated below in Figure 3.3. Without lose of generality, we only need to check that G doesn't produce extra critical points on one of them e.g. for the points

$$(x,s,t) \in \big(\partial B \times (-1,1) \times (0,2)\big) \cap \big(\mathrm{Int}\, T \times (0,2)\big),$$

where $s \in (0+\epsilon, 1)$ for some $0 < \epsilon < 1$:

$$\begin{aligned}
\frac{\partial G}{\partial t} &= \frac{\partial(\phi_1 g^+ + \phi_2 g)}{\partial t} \\
&= \phi_1 \frac{\partial g^+}{\partial t} + g^+ \frac{\partial \phi_1}{\partial t} + \phi_2 \frac{\partial g}{\partial t} + g \frac{\partial \phi_2}{\partial t} \\
&= -\phi_1 + t\frac{\partial \phi_1}{\partial t} - \sin(\pi s)\cdot \phi_2 + t\sin(\pi s)\cdot \frac{\partial \phi_2}{\partial t}
\end{aligned}$$

Here $\frac{\partial \phi_i}{\partial t} = 0$ for all $i = 1, 2, 3$ as we can define the bump functions ϕ_i independent of t, therefore,

$$\begin{aligned}
\frac{\partial G}{\partial t} &= -\phi_1 - \sin(\pi s)\cdot \phi_2 \\
&= -(\phi_1 + \phi_2) + (1 - \sin(\pi s))\phi_2 \\
&= -1 + (1 - \sin(\pi s))\phi_2 < 0.
\end{aligned}$$

So we obtain the desired G on the neighbourhood of ∂M.

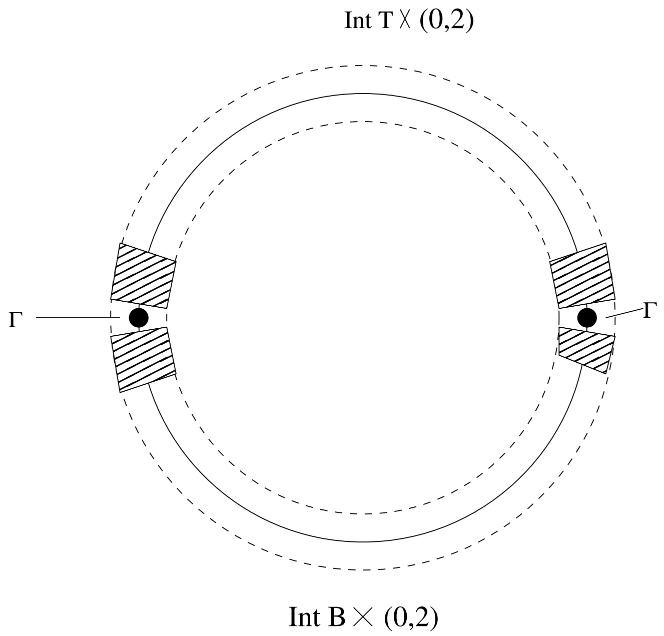

Figure 3.3: Overlaps of the open cover

Now for any given Morse function $f : \operatorname{Int} M \to \mathbb{R}$, we can glue f and G in a similar fashion using partition of unity and modify the new function to be Morse, then we have successfully constructed a function F for the lemma. $\square$

Let us state the homotopy theorem as the main result of this section:

Theorem 3.5.2 Let ω be a Morse closed 1-form on a manifold M with boundary ∂M where B is the exit set of ω, then $\varphi_v : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(\tilde{M}, \tilde{B}) \to C_*^{\mathrm{Nov}}(M, B, \omega, v)$ is a chain homotopy equivalence, i.e.

$$\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(\tilde{M}, \tilde{B}) \simeq C_*^{\mathrm{Nov}}(M, B, \omega, v).$$

The idea of the proof is to construct a cohomologous 1-form ω_1 whose gradient vector field is dominated by the gradient u of its exact component $\mathrm{d}F$, so that the chain map $\varphi_u : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^{\Delta}(\tilde{M}, \tilde{B}) \to \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(\tilde{M}, \tilde{B}, F, u)$ can be broken down to $\varphi_u = \mathrm{id} \otimes \varphi_u^{\mathrm{MS}}$ where the second map is an isomorphism proved in Chapter 1.

Proof: According to the previous lemma, we have a Morse function F sharing the same exit set B as ω, denote its gradient as u. Now we modify ω locally so that the new form ω' is constantly 0 in a small neighbourhood of each critical point of F. The idea is similar to the approximation Lemma 3.3.3. Suppose $p \in \mathrm{Crit}\, F$ is a critical point of F, then choose a contractible neighbourhood U_p of p such that there exists $f : U \to \mathbb{R}$ with $\mathrm{d}f = \omega|_{U_p}$. Restrict f to a smaller neighbourhood V_p of p, and extend such function f smoothly to the whole manifold such that the new function, say f' satisfies that $f'|_{V_p} = f$ and $f'|_{M-U_p} = 0$. See Figure 3.4 below.

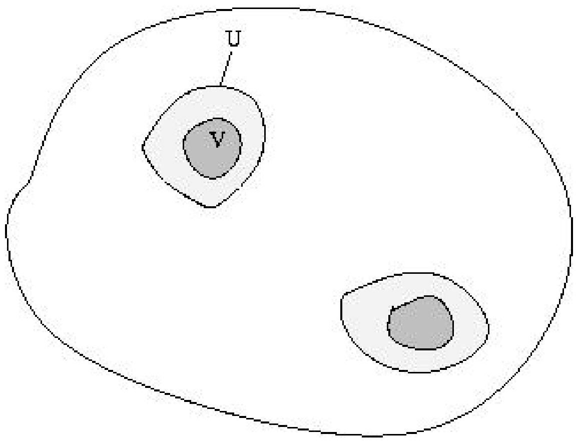

Figure 3.4: Operation near a critical point

Now add $-\mathrm{d}f'$ to ω to get $\omega - \mathrm{d}f'$, and we can see it vanishes in the neighbourhood V_p of p; repeat this operation finitely as many times as the number of the critical points of F, we obtain the desired new cohomologous 1-form ω'. Notice also that each function vanishes outside a neighbourhood of the critical point which it is responsible for, i.e. it has no impact to the boundary, and therefore the assumptions for the exit set B in particular. Now for sufficiently large constant K, the cohomologous closed 1-form $\omega_1 = \omega' + K\mathrm{d}F$ enjoys the same gradient vector field u of F, so the Novikov complex of ω_1 is really a Morse complex tensored with the Novikov ring of $\xi = [\omega]$: $C_*^{\mathrm{Nov}}(M, B, \omega_1, u) = \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(\tilde{M}, \tilde{B}, F, u)$, which has the same chain homotopy type of $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^{\Delta}(\tilde{M}, \tilde{B})$ by the chain map $\varphi_u = \mathrm{id} \otimes \varphi_u^{\mathrm{MS}}$. Now according to the continuation argument in the previous section,

$$C_*^{\mathrm{Nov}}(M, B, \omega, v) \simeq C_*^{\mathrm{Nov}}(M, B, \omega_1, u).$$

Therefore the following commutative diagram ensures the homotopy equivalence of φ_v:

$$\begin{tikzcd}
C_*^{\mathrm{Nov}}(M, B, \omega, v) \arrow[rr, "\simeq"] \arrow[dr, "\varphi_v"'] & & C_*^{\mathrm{Nov}}(M, B, \omega_1, u) \arrow[dl, "\varphi_u = \mathrm{id}\otimes\varphi_u^{\mathrm{MS}}"', "\simeq"] \\
& \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^{\Delta}(\tilde{M}, \tilde{B}) &
\end{tikzcd}$$

□

3.6 Morse inequalities

The above homotopy equivalence statement can be rephrased in the form of *Novikov Principle* for manifolds with boundary:

Theorem 3.6.1 Let ω be a Morse closed 1-form on a compact manifold M with boundary ∂M, and suppose $B \subset \partial M$ is the exit set of ω and satisfies the assumptions **A1**, **A2** and **A3**, then there exists a relative chain complex C_*^{Nov} chain homotopic to $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*(\tilde{M}, \tilde{B})$ where $G = \pi_1(M)$ is the fundamental group of M, such that for each $i = 0, \ldots, n$, C_i^{Nov} is free and finitely generated by elements of a free basis, with one-to-one correspondence with the zeros of ω of index i.

Similar to the real function case, since the number of critical points of index i equals the rank of the ith chain group, we obtain so-called *Morse-Novikov* inequalities:

Theorem 3.6.2 Let $\beta_i = \operatorname{rank} H_i(M, B; \widehat{\mathbb{Z}G_\xi})$ be the rank of $H_i(M, B; \widehat{\mathbb{Z}G_\xi})$ and $c_i = \operatorname{rank} C_i^{\mathrm{Nov}}(M, B, \omega, v)$ be the rank of $C_i^{\mathrm{Nov}}(M, B, \omega, v)$, then there exists a non-negative polynomial $R(t)$ such that

$$\sum_{i=0}^{m} t^i c_i = \sum_{i=0}^{m} t^i \beta_i + (1+t)R(t),$$

where m is the dimension of the manifold M. □

Chapter 4

The exact sequences

So far we have constructed the relative Morse-Novikov complex for a function/closed 1-form. A natural step forward is to understand its relationship with the classical absolute Morse-Novikov chain complex. In this chapter, we derive some exact sequences regarding this inquiry. We relate the relative Morse-Novikov complex and the absolute ones when looking at a compact manifold with boundary. We begin with the exact case and follow by the closed one. Most of the effort is focused on constructing a Morse function/closed 1-form with exit set satisfying assumptions $\mathbf{A1}, \mathbf{A2}$ and $\mathbf{A3}$. The chapter is concluded with some improved Morse-Novikov inequalities.

4.1 On a real Morse function

Given a compact manifold M with boundary ∂M, let us consider a 0-codimensional submanifold of ∂M, denote it B, possibly with boundary. Our goal in this section is to construct an exact sequence of Morse complexes. We first do this for the entire boundary ∂M, i.e. we consider the pair $(M, \partial M)$; and then we consider a proper submanifold $B \subset \partial M$ and the pair (M, B). The latter situation is the one we prepared in Chapter 1.

4.1.1 Short exact sequence for $(M, \partial M)$

Let $f : \partial M \to \mathbb{R}$ be a Morse function on ∂M, we want to extend f to M in such a way that it satisfies the following conditions:

1. It does not create extra critical points on a collaring neighbourhood of ∂M, namely, $\partial M \times [0, \epsilon]$ for small $\epsilon > 0$.

2. Let F be the extension of f, then the flow generated by the gradient of F exit M through ∂M in compliance with the Assumptions **A1, A2** and **A3**.

Fix a Riemannian metric, and let v be the gradient vector field of f on ∂M. To satisfy the above condition, we first want to shift the boundary one unit up along the t coordinate. Namely, we choose a collaring neighbourhood of ∂M, i.e. $\partial M \times [0, 2] \subset M$, and identify $\partial M = \partial M \times \{1\}$. Now define $f^+ : \partial M \times [0, 2] \to \mathbb{R}$ as $f^+(x, t) = f(x) + t^2 - K$ for a large positive K. Then we have the gradient v^+ of f^+ as $v^+(x, t) = (v(x), 2t\partial_t)$ with respect to the product Riemannian metric.

Suppose we have $g : M - \partial M \times [0, 1)$ as another real Morse function, we want to show firstly:

Lemma 4.1.1 There exists a Morse-Smale function F with critical points the same as the ones of f and g, and satisfies condition (1) and (2) above on $\partial M \times \{1\}$.

Proof: Define a smooth function $\rho : [0, 2] \to [0, 1]$ such that $\rho(t) = 1$ for $0 \leq t < 1 + \delta$ with $0 < \delta < \frac{1}{2}$, and $\rho(t) = 0$ for $t = 2$, in particular, $\dfrac{\partial \rho}{\partial t} < 0$ for $2 - \delta \leq t \leq 2$. We want to show that function $F = L\rho f^+ + (1 - \rho)g$ does not create any new critical points, particularly in the collaring $\partial M \times [0, 2]$. Also observe that what we need to check is essentially that the differentiation in the t coordinate is nonzero.

On $\partial M \times [0, 1]$, we have

$$\frac{\partial F}{\partial t}(x, t) = Lf^+ \frac{\partial \rho}{\partial t}(x, t) + L\rho \frac{\partial f^+}{\partial t}(x, t) = 2L \cdot \rho(t) \cdot t = 2L \cdot t = 0 \text{ iff } t = 0;$$

On $\partial M \times [1,2]$, we have

$$\begin{aligned}
\frac{\partial F}{\partial t}(x,t) &= Lf^{+}\frac{\partial \rho}{\partial t}(x,t) + L\rho\frac{\partial f^{+}}{\partial t}(x,t) + (1-\rho)\frac{\partial g}{\partial t}(x,t) - g\frac{\partial \rho}{\partial t}(x,t) \\
&= (Lf^{+} - g)(x,t)\cdot\frac{\partial \rho}{\partial t}(t) + \frac{\partial g}{\partial t}(x,t) + (L\frac{\partial f^{+}}{\partial t} - \frac{\partial g}{\partial t})(x,t)\cdot\rho(t).
\end{aligned}$$

Here

$$(L\frac{\partial f^{+}}{\partial t} - \frac{\partial g}{\partial t})(x,t)\cdot\rho(t) = (2Lt - \frac{\partial g}{\partial t}(x,t))\cdot\rho(t) \geq 0,$$

for large L as $\dfrac{\partial g}{\partial t}$ is bounded on $\partial M \times [1,2]$. Similarly,

$$(Lf^{+} - g)(x,t)\cdot\frac{\partial \rho}{\partial t}(t) + \frac{\partial g}{\partial t} = (Lf(x) + Lt^2 - LK - g(x,t))\cdot\frac{\partial \rho}{\partial t}(t) + \frac{\partial g}{\partial t} > 0,$$

for sufficiently large K, as $\dfrac{\partial \rho}{\partial t} \leq -c$ for some $c > 0$ and $\dfrac{\partial g}{\partial t} \leq C$ for some $C > 0$ by compactness. □

Remark 4.1.2 This is again a rather technical lemma, similar results are abundant. For instance, compare to [8, Lemma 1.13], where extra attention is paid to the Riemannian metric.

With the extension F constructed above, the Conditions (1) and (2) in the beginning of the chapter are readily satisfied on $\partial M \times \{1,\}$. Nevertheless, F is not automatically Smale-transverse, but the transversality condition can be restored according to the *Kupka-Smale Theorem*, see [21], [39] and [3, Chapter 7]. Using again the techniques in [29], we can perturb F slightly while keeping $f = F|_{\partial M}$ fixed so that it is transverse, then we propose a short exact sequence for the relative Morse complex:

$$0 \to C_*^{\mathrm{MS}}(\partial M, f, v_f) \to C_*^{\mathrm{MS}}(M, F, V) \to C_*^{\mathrm{MS}}(M, \partial M, g, v_g) \to 0,$$

where v_f, v_g and V are the gradient vector fields of f, g and F respectively.

Here $C_*^{\mathrm{MS}}(\partial M, f, v_f)$ is the Morse complex of f, and $C_*^{\mathrm{MS}}(M, \partial M, g, v_g)$ is the relative Morse complex following the construction in Chapter 1. The middle term $C_*^{\mathrm{MS}}(M, F, V)$ can be seen algebraically as a direct sum of the two ends, i.e. generated by the critical points of f on ∂M and g on $(M - \partial M \times [0,1), \partial M \times [0,1)) \simeq (M - \partial M \times [0,1), \partial M \times \{1\})$. We want to identify it with the simplicial complex $C_*^{\Delta}(M)$ up to chain homotopy and confirm it is indeed the Morse complex of F.

We build chain maps $\varphi_{v_f}^{\partial M} : C_*^{\Delta}(\partial M) \to C_*^{\mathrm{MS}}(\partial M, f, v_f)$ and $\varphi_{v_g}^{(M,\partial M)} : C_*^{\Delta}(M, \partial M) \to C_*^{\mathrm{MS}}(M, \partial M, g, v_g)$ as in Chapter 1. Namely, we choose triangulations adjusted to v_f and v_g respectively, this firstly induces the chain map φ_{v_f}. Now for $\varphi_{v_g}^{(M,\partial M)} : C_*^{\Delta}(M, \partial M) \to C_*^{\mathrm{MS}}(M, \partial M, g, v_g)$, since the flow of F exit $\partial M \times \{1\}$ transversely, any triangulation of $\partial M \times \{1\}$ is automatically adjusted to v_g in $M - \partial M \times [0, 1)$, so $\varphi_{v_g}^{(M,\partial M)}$ is well-defined too.

Now we want to define a chain map between $C_*^{\Delta}(M)$ and $C_*^{\mathrm{MS}}(M, F, V)$. We choose a common subdivision Δ^s of the above triangulations adjusted to both v_f and v_g. Notice the unstable manifolds of the critical points in ∂M come from the whole manifold, in particular, we need to reconsider the way we define incidence number $[\sigma : p]$ for $\sigma \in \Delta^s$ and $p \in \operatorname{Crit} F|_{\partial M}$. Now since ∂M is a subcomplex of M, we can talk about two cases, namely, the simplices from ∂M and the simplices from $M - \partial M$. For $\sigma \in \Delta^s|_{\partial M}$, $[\sigma : p]$ is defined as in the manifold ∂M with the Morse function f; and for $\sigma \in \Delta^s|_{M - \partial M}$, we first slide σ alone the flow going towards ∂M and then compare the intersections $\sigma \cap W^u(p, V) \subset M$ with the oriented stable manifold $W^s(p, V) \subset M$. Then we get $[\sigma : p]$ and hence define φ_V^M.

Now consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_*^{\Delta^s}(\partial M) \arrow[r] \arrow[d, "\varphi_{v_f}^{\partial M}"] & C_*^{\Delta^s}(M) \arrow[r] \arrow[d, "\varphi_V^M"] & C_*^{\Delta^s}(M, \partial M) \arrow[r] \arrow[d, "\varphi_{v_g}^{(M,\partial M)}"] & 0 \\
0 \arrow[r] & C_*^{\mathrm{MS}}(\partial M, f, v_f) \arrow[r] & C_*^{\mathrm{MS}}(M, F, V) \arrow[r] & C_*^{\mathrm{MS}}(M, \partial M, g, v_g) \arrow[r] & 0
\end{tikzcd}$$

Since $\varphi_{v_f}^{\partial M}$ and $\varphi_{v_g}^{(M,\partial M)}$ are chain homotopy equivalence, the Five Lemma shows φ_V^M is a chain homotopy equivalence too. Therefore, we indeed have a short exact sequence for the relative Morse complexes:

Theorem 4.1.3 Let M be a compact manifold with boundary ∂M. Suppose we also have Morse functions $f : \partial M \to \mathbb{R}$ and $g : M - \partial M \times [0, 1)$, then there exists Morse function F with $F|_{\partial M} = f$ and $F|_{M - \partial M \times [0,1)} = g$, such that we have the following short exact sequence of Morse complexes:

$$0 \to C_*^{\mathrm{MS}}(\partial M, f, v_f) \to C_*^{\mathrm{MS}}(M, F, V) \to C_*^{\mathrm{MS}}(M, \partial M, g, v_g) \to 0.$$

Remark 4.1.4 In fact, by the time we finished the development of this exact sequence for Morse complexes, we noticed [1] has the results already published. Nevertheless, we are aware of our construction of an explicit Morse function for the whole manifold. Both approaches are based on a cobordism setting, namely, in our context, the boundary Γ of the exit set $\partial M \times \{1\}$ is empty.

4.1.2 Another short exact sequence

Let B be a 0-codimensional compact submanifold of ∂M and let $f : \partial M \to \mathbb{R}$ be a Morse function on the boundary ∂M so that $f^{-1}((-\infty, 0]) = B$. Note that the boundary of B needs not to be empty, in which case we denote it Γ, then $T = \overline{\partial M - B}$ has boundary $\partial T = \Gamma$ too. Γ is a 1-codimensional submanifold of ∂M, and we have tubular neighbourhood $\Gamma \times [-1, 1]$ of Γ with a chosen orientation so that $\dfrac{\partial f}{\partial s}(x, 0) > 0$ for $(x, 0) \in \Gamma \times [-1, 1]$ where s is the coordinate of $[-1, 1]$. This construction is possible as we can always perturb f slightly in ∂M so that f has no critical points on Γ, and then approximate $f|_{\Gamma\times[-1,1]}$ by projection to the last coordinate $[-1, 1]$.

Now consider pair (M, B), since f restricted to B is still Morse and if we define $C_*(M, B, F, V)$ to be the quotient of the two Morse complexes $C_*^{\mathrm{MS}}(B, f, v_f)$ and $C_*^{\mathrm{MS}}(M, F, V)$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_*^{\Delta^s}(B) \arrow[r] \arrow[d, "\simeq"] & C_*^{\Delta^s}(M) \arrow[r] \arrow[d, "\simeq"] & C_*^{\Delta^s}(M, B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*^{\mathrm{MS}}(B, f, v_f) \arrow[r] & C_*^{\mathrm{MS}}(M, F, V) \arrow[r] & C_*(M, B, F, V) \arrow[r] & 0
\end{tikzcd}$$

So by Five Lemma, $C_*(M, B, F, V)$ is chain homotopically identified with $C_*^{\Delta^s}(M, B)$. We want to relate this chain complex to our construction in Chapter 1. Namely, adapt B as the exit set of a modified Morse function $\tilde{F}$.

Firstly, We enlarge the manifold M similarly to the construction in the beginning of Chapter 1, and rescale the tubular neighbourhood of ∂M in the previous section as $\partial M \times [-2, 4]$ with $\partial M \cong \partial M \times \{0\}$, so that $F|_{\partial M\times[-2,2]}(x, t) = f^+(x, t) = f(x) + t^2 - K$.

Then what we want to do now is to find an embedding $i : \partial M \times [-1, 1] \hookrightarrow \partial M \times [-2, 2]$ so that the composition $f^+ i : \partial M \times [-1, 1] \to \mathbb{R}$ satisfies assumptions **A1**, **A2** and **A3**, and also recognises B as the exit set. We do this in two steps:

Step one, let $a : \partial M \to [-1,1]$ be a smooth function so that $0 \le a(x) \le 1$ for $x \in B$ and $-1 \le a(x) \le 0$ for $x \in T$, and $a(x) = 0$ for $x \in \Gamma$ and 0 is a regular value for a. Then we have an embedding $\partial M \to \partial M \times [-1,1]$ as $x \mapsto (x, a(x))$. Note that for the exact case at this stage, a can be chosen as a multiple of f adjusted to sign.

Step two, we define $i : \partial M \times [-1,1] \to \partial M \times [-2,2]$ as $i(x,t) = (x, a(x)+t)$. Now we claim $f^+ i(x,t) = f(x) + \big(a(x)+t\big)^2 - K$ on $\partial M \times [-1,1]$ satisfies assumptions **A1**, **A2** and **A3** on ∂M with exit set B.

For Assumption **A1**, we want to show $f^+ i$ has no critical points on ∂M:

$$d(f^+ i)|_{(x,0)} = \begin{pmatrix} \frac{\partial f}{\partial x}\mathrm{d}x + 2a\frac{\partial a}{\partial x}\mathrm{d}x \\ 2a\mathrm{d}t \end{pmatrix} \neq 0$$

This is the case as we chose $a(p) \neq 0$ nontrivial at any critical point $p \in \operatorname{Crit} f$.

For assumption **A2**, we want zero to be a regular value of $\frac{\partial (f^+ i)}{\partial t}|_{(x,0)}$, i.e. a needs to be regular at any $x \in \partial M$ with $a(x) = 0$. This is again taken care of by the definition of a.

For assumption **A3**, we want for any point $(x,0,0) \in \Gamma \times [-1,1] \times [-2,2]$,

$$\frac{\partial (f^+ i)}{\partial s}(x,0,0) > 0.$$

Now $f^+ i(x,0,0) = f(x,0,0) + a^2(x,0,0) - K$ and

$$\frac{\partial (f^+ i)}{\partial s}(x,0,0) = \frac{\partial f}{\partial s}(x,0,0) + 2a(x,0,0)\frac{\partial a}{\partial s}(x,0,0) = \frac{\partial f}{\partial s}(x,0,0) > 0,$$

for we chose f to be so.

Finally, by the construction of a, $\frac{\partial (f^+ i)}{\partial t} = 2a + 2t$ is always positive on $\operatorname{Int} B$ and 0 on Γ and negative on $\operatorname{Int} T$. So modify function F on M to $\tilde{F}$ with $\tilde{F}|_{\partial M \times [-1,1]} = f^+ i$, then $\tilde{F}$ has exit set B, and $C_*(M,B,F,V)$ can be viewed as the Morse complex of $\tilde{F}$:

$$C_*(M,B,F,V) = C_*^{\mathrm{MS}}(M,B,\tilde{F},V).$$

Therefore we have the following theorem for the pair (M,B):

Theorem 4.1.5 Let M be a smooth manifold with boundary ∂M, Let $B \subset \partial M$ be a 0-codimensional submanifold of ∂M, such that a Morse function $f : \Gamma \to \mathbb{R}$ is regular on its boundary $\Gamma \subset B$. Then we can perturb f slightly such that there exists Morse function $F : M \to \mathbb{R}$ with $F|_B = f$ and $F|_{M-\partial M\times[0,1)} = g$ for some Morse function $g : M - \partial M \times [0,1) \to \mathbb{R}$, and B is the exit set of F in the sense of assumptions **A1**, **A2** and **A3**, and we have the following exact sequence:

$$0 \to C_*^{\mathrm{MS}}(B, f, v_f) \to C_*^{\mathrm{MS}}(M, F, V) \to C_*^{\mathrm{MS}}(M, B, F, V) \to 0.$$

□

4.1.3 Homology and improved Morse inequalities

The short exact sequence in the previous section induces a long exact sequence:

Corollary 4.1.6 With the above settings for the pair (M, B), we have the following long exact sequence of their Morse homology:

$$\cdots \xrightarrow{\partial_{k+1}} H_k^{\mathrm{MS}}(B) \xrightarrow{i_k} H_k^{\mathrm{MS}}(M) \xrightarrow{j_k} H_k^{\mathrm{MS}}(M, B) \xrightarrow{\partial_k} H_{k-1}^{\mathrm{MS}}(B) \xrightarrow{i_{k-1}} \cdots ,$$

where $H_*^{\mathrm{MS}}(B), H_*^{\mathrm{MS}}(M)$ and $H_*^{\mathrm{MS}}(M, B)$ are the Morse homology of B, M and (M, B) by functions f, F and $F|_{M-\partial M\times[0,1)} = g$, respectively. □

Now for each k, let $\mathrm{rank}\,(\partial_{k+1})$ be the dimension of the image of ∂_{k+1} and $\beta_k(B)$ be the kth Betti number of B, and so on. Then notice

$$\mathrm{rank}\,(\partial_{k+1}) = \beta_k(B) - \mathrm{rank}\,(i_k)$$

and

$$\mathrm{rank}\,(i_k) = \beta_k(M) - \mathrm{rank}\,(j_k).$$

Because $\mathrm{rank}\,(\partial_{k+1}) \geq 0$, iterate it finitely many times, we get

$$\sum_{i=0}^{k}(-1)^{k-i}\beta_i(B) - \sum_{i=0}^{k}(-1)^{k-i}\beta_i(M) + \sum_{i=0}^{k}(-1)^{k-i}\beta_i(M, B) \geq 0,$$

hence

$$\sum_{i=0}^{k}(-1)^{k-i}\beta_i(M, B) + \sum_{i=0}^{k}(-1)^{k-i}\beta_i(B) \geq \sum_{i=0}^{k}(-1)^{k-i}\beta_i(M)$$

for all k, and

$$\sum_{i=0}^{m}(-1)^{m-i}\beta_i(M, B) + \sum_{i=0}^{m}(-1)^{m-i}\beta_i(B) = \sum_{i=0}^{m}(-1)^{m-i}\beta_i(M),$$

where m is the dimension of M.

We summarise these inequalities in the following polynomial:

Corollary 4.1.7 There exists a non-negative polynomial $R(t)$ such that

$$\sum_i t^i\big(\beta_i(M,B)+\beta_i(B)\big) \;=\; \sum_i t^i\beta_i(M)+(1+t)R(t).$$

□

Notice also for the Morse function F constructed in the preceding section, we can improve the estimate of its critical points as follows:

$$\begin{aligned}\sum_{i=0}^{k}(-1)^{k-i}\beta_i(M) &\le \sum_{i=0}^{k}(-1)^{k-i}\beta_i(M,B)+\sum_{i=0}^{k}(-1)^{k-i}\beta_i(B)\\ &\le \sum_{i=0}^{k}(-1)^{k-i}c_i(M,B)+\sum_{i=0}^{k}(-1)^{k-i}c_i(B) \qquad (4.1)\\ &= \sum_{i=0}^{k}(-1)^{k-i}c_i(M). \qquad (4.2)\end{aligned}$$

Here (4.1) is the standard results of Morse inequalities for function F and f on M and B respectively; whereas (4.2) comes from the fact that the number of critical points of F on M is the sum of number of critical points F on B and on M away from B.

Therefore, we have the following improved Morse inequalities:

Corollary 4.1.8 There exists a non-negative polynomial $R(t)$ such that

$$\sum_{i=0}^{m} t^i c_i(M) \;=\; \sum_{i=0}^{m} t^i\big(\beta_i(M,B)+\beta_i(B)\big)+(1+t)R(t),$$

where m is the dimension of the manifold M. □

We provide a simple example in the followings to conclude this subsection:

Example 4.1.9 Consider a torus T^- with two points deleted, hence with boundary $\partial T^- = S^1\sqcup S^1$, depicted in Figure 4.1 below.

Let f be the high function assign to T^-. Then we have critical points p_1,p_2,p_3 and p_4 in the interior, a_1,a_2 and b_1,b_2 on the boundary circles. Note that a_1 and b_1 have index 0, a_2,b_2,p_1,p_2 and p_3 have index 1, whereas p_4 has index 2. Now a simple calculation shows us the Betti numbers of T^-: $\beta_0(T^-)=1$,$\beta_1(T^-)=3$ and

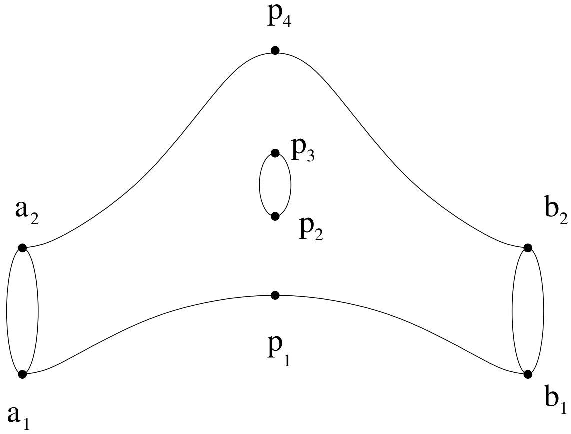

Figure 4.1: Torus with boundary

$\beta_2(T^-) = 0$. On the other hand, $\beta_0(\partial T^-) = \beta_1(\partial T^-) = 2$ and $\beta_2(\partial T^-) = 0$; and $\beta_0(T^-, \partial T^-) = 0$, $\beta_1(T^-, \partial T^-) = 3$ and $\beta_2(T^-, \partial T^-) = 1$. Hence we have a sharper lower bound for the critical points (in fact in this case equalities):

$$c_0 = 2 = \beta_0(T^-, \partial T^-) + \beta_0(\partial T^-) > \beta_0(T^-) = 1,$$

$$c_1 = 5 = \beta_1(T^-, \partial T^-) + \beta_1(\partial T^-) > \beta_1(T^-) = 3,$$

and

$$c_2 = 1 = \beta_2(T^-, \partial T^-) + \beta_2(\partial T^-) > \beta_2(T^-) = 0.$$

□

4.2 On a closed 1-form

Let Ω be a Morse closed 1-form on M with boundary ∂M, and let the flow of the gradient grad Ω exit through ∂M with ∂M satisfying assumptions **A1**, **A2** and **A3** (In fact, when the exit set is the whole ∂M, **A3** is redundant as $\Gamma = \emptyset$). Further more, suppose ω is a closed 1-form with its cohomology class $[\omega] = i^*([\Omega]) \in H^1(\partial M)$. Here $i^* : H^1(M) \to H^1(\partial M)$ is the cohomology map induced by the inclusion $i : \partial M \to M$. We want to show the construction of such closed 1-form so that we get a short exact sequence of the Novikov complexes analogous to the one we constructed above from a real Morse function. We first suggest a slightly rough version here:

$$0 \to C_*^{\text{Nov}}(\partial M, \omega) \to C_*^{\text{Nov}}(M, \Omega) \to C_*^{\text{Nov}}(M, \partial M, \Omega) \to 0$$

Now we make this precise:

Theorem 4.2.1 Given a closed 1-form Ω' on a manifold M with boundary, we can always modify Ω' slightly within its cohomology class $[\Omega'] \in H^1(M)$ so that the modified closed 1-forms Ω on M and ω on ∂M derive the Novikov complexes as described above respectively, and they form a short exact sequence.

Proof: Let Ω' be a closed 1-form on M. Consider the cohomology class $[\Omega'] \in H^1(M)$ of Ω', then we have its image $i^*([\Omega']) \in H^1(\partial M)$ under the induced map $i^* : H^1(M) \to H^1(\partial M)$. Choose a Morse closed 1-form ω on ∂M such that $[\omega] = i^*([\Omega'])$, and let $\partial M \times [0, 2]$ be the collaring of ∂M with $\partial M \times \{0\} \cong \partial M$. We extend ω to $\omega + dt^2$ on $\partial M \times [0, 2]$. On the other hand, if we denote the restriction of Ω' on $\partial M \times [0, 2]$ as $\omega' = \Omega'|_{\partial M \times [0,2]}$, there is a function $f : \partial M \times [0, 2] \to \mathbb{R}$ such that $\omega' + df = \omega + dt^2$. Extend f trivially to the whole M, say F, such that $F|_{\partial M \times [0,1]} = f$ and $F|_{M - \partial M \times [0,2)} = 0$ then we have $\Omega' + dF|_{\partial M \times [0,1]} = \omega + dt^2$ which is Morse. Next, we perturb the closed 1-form $\Omega' + dF$ on neighbourhoods of its critical sets so it becomes nondegenerate without changing its cohomology class. This is possible for $\Omega' + dF$ is exact on each neighbourhood and we can add an exact form with coefficient large enough so that the stable and unstable manifolds intersect transversely.

We denote this modification Ω, notice $\Omega|_{\partial M \times \{1\}} = \omega + dt^2$ has gradient vector field in the form $(\operatorname{grad} \omega, 2t\partial_t)$, and its flow exits $\partial M \times \{1\}$ transversely, so

$$C_*^{\text{Nov}}(\partial M \times [0, 1], \omega + dt^2) \simeq C_*^{\text{Nov}}(\partial M, \omega)$$

and

$$C_*^{\text{Nov}}(M, \partial M \times [0, 1], \Omega) \simeq C_*^{\text{Nov}}(M \setminus \partial M \times [0, 1), \partial M \times \{1\}, \Omega)$$

We also employ Smale transversality approximation on the gradient vector field of Ω and ω, then the constructions of the three Novikov complexes are complete.

Now $C_*^{\text{Nov}}(\partial M, \omega) = \widehat{\mathbb{Z}H}_\eta \otimes C_*(\widetilde{\partial M}, \omega)$, where $H = \pi_1(\partial M), \eta = [\omega]$ and $C_*(\widetilde{\partial M}, \omega)$ is a $\mathbb{Z}H$ module generated by the zeros of ω. Similarly, $C_*^{\text{Nov}}(M, \Omega) =$

$\widehat{\mathbb{Z}G}_\xi \otimes C_*(\tilde{M}, \Omega)$, where $G = \pi_1(M), \xi = [\Omega]$ and $C_*(\tilde{M}, \Omega)$ is a $\mathbb{Z}G$ module generated by the zeros of Ω.

Consider the map $i_* : \pi_1(\partial M) \to \pi_1(M)$ induced by inclusion $i : \partial M \to M$. By slightly abusing the notation, we have a natural group ring homomorphism $i_* : \mathbb{Z}H \to \mathbb{Z}G$, therefore, $\widehat{\mathbb{Z}G}_\xi$ can be seen as a module of $\widehat{\mathbb{Z}H}_\eta$ and we can tensor $C_*^{\text{Nov}}(N, \omega)$ with $\widehat{\mathbb{Z}G}_\xi$ to get $\widehat{\mathbb{Z}G}_\xi \otimes_{\widehat{\mathbb{Z}H}_\eta} C_*^{\text{Nov}}(\partial M, \omega)$.

The last thing we need to show now is that the map

$$\text{id} \otimes j_* : \widehat{\mathbb{Z}G}_\xi \otimes_{\widehat{\mathbb{Z}H}_\eta} C_*^{\text{Nov}}(\partial M, \omega) \to C_*^{\text{Nov}}(M, \Omega)$$

is indeed a chain map, where $j_* : C_*(\widetilde{\partial M}, \omega) \to C_*(\tilde{M}, \Omega)$ is induced from the inclusion $j : \partial M \to M$.

In other words, we want to show the existence of the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathbb{Z}G}_\xi \otimes C_q(\widetilde{\partial M}, \omega) \arrow[r, "\text{id}\otimes j_*"] \arrow[d, "\text{id}\otimes\partial_{\partial M}"'] & \widehat{\mathbb{Z}G}_\xi \otimes C_q(\tilde{M}, \Omega) \arrow[d, "\text{id}\otimes\partial_M"] \\
\widehat{\mathbb{Z}G}_\xi \otimes C_{q-1}(\widetilde{\partial M}, \omega) \arrow[r, "\text{id}\otimes j_*"] & \widehat{\mathbb{Z}G}_\xi \otimes C_{q-1}(\tilde{M}, \Omega)
\end{tikzcd}$$

Notice the chain complex $C_*(\tilde{M}, \Omega)$ consists of two components, namely,

$$C_*(\tilde{M}, \Omega) = C_*(\widetilde{\partial M}, \omega) \oplus C_*(\tilde{M} \setminus \widetilde{\partial M} \times [0, 1), \Omega)$$

so for $\sigma \in C_q(\widetilde{\partial M}, \omega)$, $j_*(\sigma) = (\sigma, 0) \in C_q(\tilde{M}, \Omega)$, and

$$\partial_M = \begin{pmatrix} \partial_{\partial M} & \phi_{M,\partial M} \\ 0 & \partial_{M\setminus\partial M} \end{pmatrix},$$

here the bottom left entry is trivial because the flow of gradΩ exits out from ∂M in one direction, i.e. the trajectories come from zeros of ω in ∂M stay in ∂M.

Now it is an easy exercise to show that $\partial_M \cdot j_*(\sigma) = \partial_M(\sigma, 0) = (\partial_{\partial M}\sigma, 0) = j_* \cdot \partial_{\partial M}(\sigma)$.

Finally, we have the desired sequence over the Novikov ring $\widehat{\mathbb{Z}G}_\xi$:

$$0 \to \widehat{\mathbb{Z}G}_\xi \otimes_{\widehat{\mathbb{Z}H}_\eta} C_*^{\mathrm{Nov}}(\partial M, \omega) \to C_*^{\mathrm{Nov}}(M, \Omega) \to C_*^{\mathrm{Nov}}(M, \partial M, \Omega) \to 0.$$

□

Following a five lemma type argument, we find chain homotopy equivalence between each component of the sequence and the (relative) simplicial complex of the underlying manifold:

Theorem 4.2.2 The following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathbb{Z}G}_\xi \otimes_{\widehat{\mathbb{Z}H}_\eta} C_*^{\mathrm{Nov}}(\partial M, \omega) \arrow[r] \arrow[d, "\simeq"] & C_*^{\mathrm{Nov}}(M, \Omega) \arrow[r] \arrow[d, "\simeq"] & C_*^{\mathrm{Nov}}(M, \partial M, \Omega) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \widehat{\mathbb{Z}G}_\xi \otimes C_*^{\Delta}(\partial M) \arrow[r] & \widehat{\mathbb{Z}G}_\xi \otimes C_*^{\Delta}(M) \arrow[r] & \widehat{\mathbb{Z}G}_\xi \otimes C_*^{\Delta}(M, \partial M) \arrow[r] & 0
\end{tikzcd}.$$

□

Now consider a 0-codimensional compact submanifold B of ∂M, possibly with boundary. Then choose a closed 1-form Ω in the cohomology class ξ, such that its restriction on B satisfies the boundary conditions $\mathbf{A1}, \mathbf{A2}$ and $\mathbf{A3}$. Namely, we want $\omega = \Omega|_B$ to be free of critical points on Γ, where Γ is the boundary of B, and

$$-\frac{\partial f}{\partial t}(x, 0) > 0,$$

for $(x, 0) \in \Gamma \times [0, 1) \subset B$ and f is the lift of ω on the covering space with partial derivative $\dfrac{\partial f}{\partial t}$ defined on B equivariantly. In other words, ω has only entry set but no exit set on B, and therefore the Novikov complex $C_*^{\mathrm{Nov}}(B, \omega)$ of B under ω is consistent with our construction in Chapter 3.

Theorem 4.2.3 Let B be a 0-codimensional compact submanifold of ∂M, possibly with boundary. Then we get a similar short exact sequence for the pair (M, B) where B is seen as the exit set of a closed 1-form Ω with $\Omega|_B = \omega$:

$$0 \to \widehat{\mathbb{Z}G}_\xi \otimes C_*^{\mathrm{Nov}}(B, \omega) \to C_*^{\mathrm{Nov}}(M, \Omega) \to C_*^{\mathrm{Nov}}(M, B, \Omega) \to 0.$$

Consider the map $\widehat{\mathbb{Z}H}_\eta \twoheadrightarrow \mathbb{Z}[\widehat{H/\ker\eta}]_\eta \hookrightarrow \mathbb{Q}[\widehat{H/\ker\eta}]_\eta$ and the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathbb{Z}H}_\eta & \longrightarrow & \widehat{\mathbb{Z}G}_\xi \\ \downarrow & & \downarrow \\ \mathbb{Z}[\widehat{H/\ker\eta}]_\eta & \longrightarrow & \mathbb{Q}[\widehat{G/\ker\xi}]_\xi \end{array}$$

where $\mathbb{Z}[\widehat{H/\ker\eta}]_\eta \to \mathbb{Q}[\widehat{G/\ker\xi}]_\xi$ factors through $\mathbb{Q}[\widehat{H/\ker\eta}]_\eta$.

Use shorthand notation $\mathbb{Q}(\xi) = \mathbb{Q}[\widehat{G/\ker\xi}]_\xi$. Now since $\mathbb{Q}(\xi)$ is a field, after tensoring with the Novikov complexes in the previous exact sequence, we have the following short exact sequence:

$$0 \to \mathbb{Q}(\xi) \otimes C_*^{\mathrm{Nov}}(B,\omega) \to \mathbb{Q}(\xi) \otimes C_*^{\mathrm{Nov}}(M,\Omega) \to \mathbb{Q}(\xi) \otimes C_*^{\mathrm{Nov}}(M,B,\Omega) \to 0.$$

This in turns induces a long exact sequence of vector spaces:

Corollary 4.2.4 With the above notations, we have a long exact sequence of homology of Novikov complexes:

$$\cdots \to H_k^{\mathrm{Nov}}(B;\mathbb{Q}(\xi)) \to H_k^{\mathrm{Nov}}(M;\mathbb{Q}(\xi)) \to H_k^{\mathrm{Nov}}(M,B;\mathbb{Q}(\xi)) \to \cdots,$$

where $H_k^{\mathrm{Nov}}(B;\mathbb{Q}(\xi))$, $H_k^{\mathrm{Nov}}(M;\mathbb{Q}(\xi))$ and $H_k^{\mathrm{Nov}}(M,B;\mathbb{Q}(\xi))$ are the Novikov homology of M, B and (M,B) with coefficients in $\mathbb{Q}(\xi)$, respectively. □

So we also have improved analogous results on Morse inequalities for the relative Novikov complexes:

Corollary 4.2.5 There exists a non-negative polynomial $R(t)$ such that

$$\sum_{i=0}^m t^i\big(\beta_i(M,B)+\beta_i(B)\big) \;=\; \sum_{i=0}^m t^i\beta_i(M) + (1+t)R(t),$$

where m is the dimension of the manifold M. Here $\beta_i(M), \beta_i(B)$ and $\beta_i(M,B)$ are the rank of $H_k^{\mathrm{Nov}}(M;\mathbb{Q}(\xi)), H_k^{\mathrm{Nov}}(B;\mathbb{Q}(\xi))$ and $H_k^{\mathrm{Nov}}(M,B;\mathbb{Q}(\xi))$ respectively. □

Corollary 4.2.6 With the notations in the previous lemma and denote $c_i(M,B)$ the rank of $C_*^{\mathrm{Nov}}(M,B,\Omega)$, then there exists a non-negative polynomial $R(t)$ such that

$$\sum_{i=0}^m t^i c_i(M) \;=\; \sum_{i=0}^m t^i\big(\beta_i(M,B)+\beta_i(B)\big) + (1+t)R(t),$$

where m is the dimension of the manifold M. □

Chapter 5

Morse-Bott nondegeneracy

In this chapter, we generalise the nondegeneracy condition in the sense of Bott and direct the interests to the homology of the underlying chain complexes. On a manifold with boundary, instead of looking at the topology of isolated critical points of a function, we study its critical set as a union of connected submanifolds where the gradient vector field of the function vanishes under Bott's assumptions. While constructing the so-called Morse-Bott complex of a real function and the more generalised Novikov-Bott complex, we keep an eye towards their homology groups and finally obtain the Morse inequalities by a spectral sequence argument.

5.1 The settings

Let $f : M \to \mathbb{R}$ be a real function on manifold M with boundary ∂M, f is called *non-degenerate in the sense of Bott* if its critical set $\mathbf{C} = \mathrm{Crit}\,(f)$ satisfies the followings:

B1 The set of critical points $\mathbf{C}$ is a submanifold of M, called *critical manifold.* The critical manifold is usually not connected, so $\mathbf{C}$ often is a union of connected components of submanifolds of M, $\mathbf{C} = \bigcup_i C_i$.

B2 Denote $\nu(\mathbf{C})$ as the normal bundle of the critical manifold, then the restriction $\hat{f}_{**}$ of the Hessian f_{**} on the normal bundle $\nu(\mathbf{C})$ is pointlessly nondegenerate in $\mathbf{C}$, i.e. $\hat{f}_{**} : \big(T_pM/T_p\mathbf{C}\big) \times \big(T_pM/T_p\mathbf{C}\big) \to \mathbb{R}$ is nondegenerate for every

$p \in \mathbf{C}$.

A function which satisfies assumptions **B1** and **B2** is called *Morse-Bott function*.

According to assumption **B2**, for each connected component C of $\mathbf{C}$, the number of negative eigenvalues of the Hessian f_{**} is fixed for all points in C, this defines the index of C , denoted as $\operatorname{ind}(C)$. Let v be a gradient vector field of f with respect to some Riemannian metric, suppose it also satisfies the Smale-transversality condition, we define the *stable manifold* $W^s(C, v)$ and *unstable manifold* $W^u(C, v)$ as follows, respectively:

$$W^s(C, v) = \{x \in M : x \cdot t \to C \text{ as } t \to +\infty\},$$

$$W^u(C, v) = \{x \in M : x \cdot t \to C \text{ as } t \to -\infty\}.$$

Here $x \cdot t$ is a shorthand notation of the image of the gradient flow $\Phi : M \times \mathbb{R} \to M$, generated by the gradient v with $x \cdot 0 = x$ for each x and t.

Consider a k-dimensional connected component $C \subset \mathbf{C}$ of the critical submanifold $\mathbf{C}$ with index $\operatorname{ind}(C) = \lambda$, then its stable manifold $W^s(C, v)$ is of $k + \lambda$ dimension. Similarly for the unstable manifold. Therefore, even for some other l-dimensional component C' with index $\operatorname{ind}(C') = \mu$ such that $\mu > \lambda$, it is still possible that $W^u(C', v) \cap W^s(C, v) \neq \emptyset$, when $k + \lambda$ is sufficiently large, say $k + \lambda \geq l + \mu$ to be precise. This will cause problem when it comes to the construction of Novikov complex for closed 1-forms, namely, the trajectories of v can flow back to the same connected component of the critical manifold, in which case it is not possible to construct a filtration for the manifold as we do in the real function case. To resolve the problem, we introduce the *no homoclinic cycle* condition, namely, the existence of a gradient vector field without homoclinic cycles in M. This roughly is to exclude the possibility that a trajectory returns to the same connected component which prevents us from getting a spectral sequence for the homology of M.

We will make this condition precise in the closed 1-form situation in the corresponding section.

The definition for the exit set $B \subset \partial M$ and the transversality assumptions **A1**, **A2**, **A3** are unchanged for both Morse functions and closed 1-forms.

5.2 Spectral sequences and the chain complexes

In this section, we consider both cases: real functions and circle-valued functions which represent closed 1-forms. For each case, our approach is to construct a filtered complex from a Morse-Bott complex, and describe its boundary map by a Morse approximation, so that some spectral sequence can be derived from this filtered complex. Finally, the spectral sequence yields the homology groups of the manifold due to its convergence.

5.2.1 For real functions

There are several different filtrations for a real Morse-Bott complex, we refer to [18], [3] and [16]. A straightforward filtration can be based on the value of each connected component under f, however, this approach requires more restrictions (the self-indexing condition) on the destination of trajectories in the closed 1-form situation. Therefore, we adopt a slightly more sophisticated approach which is consistent with the *no homoclinic cycle* condition in the closed 1-form case.

The ordering

Namely, we order the connected components according to the destination of the flow. Consider the set of connected components $\{C_i\}$ of $\mathbf{C}$ as a set of vertices $\{v_i\}$, each vertex v_i corresponds to one component C_i, and we connect two vertices v_i, v_j with an directed edge $e_{ij}(e_{ji})$ if there are trajectories going from C_i to C_j (or vice versa), the direction is towards where the trajectories point. Then we obtain a directed graph without loops. We perform the following algorithm to get an ordering:

Notice the number of components are finite due to the compactness of the manifold, so we do the following algorithm by finite repetition, in round j where $j = 0, \ldots, n$:

1. For a vertex, check whether there are incident edges pointing out, label it j if there is not, otherwise repeat the check for the successive vertices where the edges are pointing to.

2. Having exhausted all the vertices by the above operation, collect the labelled ones in a union $C(j)$, delete them together with any incident edges from the graph, and start it over again as round $j+1$.

We end up having an empty graph after finitely many rounds, say n, and each vertex is assigned uniquely to a specific order, for instance, $C(j)$ is the union of the components with order j. Let us formalise this notation:

Notation 5.2.1 For each j, suppose we reindex all the connected components with order j as C_α with $\alpha \in \mathcal{A}(j)$, where $\mathcal{A}(j)$ is the collection of all the new indices of the connected components with order j. Then we denote $C(j)$ as the union of all the connected components with order j according to the above algorithm:

$$C(j) = \bigcup_{\alpha \in \mathcal{A}(j)} C_\alpha.$$

Morse-Bott chain complex and its filtration(chain inclusion map)

Now we want to define the *Morse-Bott complex* and build a spectral sequence according to the hierarchy of the connected components, which is convergent to the homology groups of M.

A direct way of doing this is to define the chain complex as a direct sum of chain complexes of the critical manifolds and then specify the boundary maps, but to build on the previous chapters, we first approximate the original function f by a generic Morse function with respect to the ordering, and then define the chain complex with boundary maps from counting trajectories in a standard way.

Consider a connected component C_i of the critical manifold $\mathbf{C}$ in M, where f is the Morse-Bott function on M, assign C_i a generic Morse function $f_i : C_i \to \mathbb{R}$, i.e. the gradient v_i of f_i with respect to some Riemannian metric is transverse, then we have a Morse complex $C_*^{\mathrm{MS}}(C_i, f_i, v_i)$ of f_i on C_i as

$$C_*^{\mathrm{MS}}(C_i, f_i, v_i) = \bigoplus_{p \in \mathrm{Crit}\,(f_i)} \mathbb{Z},$$

generated by the critical points of f_i. Having done this to every connected component C_i of $\mathbf{C}$, we want to modify the original f out of these Morse functions f_i's, so that we can eventually pin down the trajectories from one critical component to another to specific critical points of f_i's as we did in the real Morse function case of Chapter 1. Please see [2] and [18] for similar operations.

Firstly, choose a tubular neighbourhood $N(C_i)$ of C_i, then if $z \in C_i$ is a point in this critical component, we write a point in $N(C_i)$ as $(z, x_1, \ldots, x_\lambda, y_{\lambda+1}, \ldots, y_{m-\dim(C_i)}) = (z, x, y) \in N(C_i)$, where m is the dimension of M and $\lambda = \mathrm{ind}\,(C_i)$ is the index of the critical component C_i. Let $f_i^+(z, x, y) = f_i(z) - x^2 + y^2 : N(C_i) \to \mathbb{R}^+$, and choose a bump function $\rho_i : M \to [0, 1]$ such that

$$\rho_i|_{C_i} = 1 \text{ and } \rho_i|_{M-N(C_i)} = 0.$$

Then function $F_i = f + \epsilon\rho_i f_i^+$ creates no extra critical points other than the ones of f_i and f, for ρ_i with small partial derivatives $\dfrac{\partial \rho_i}{\partial x}$:

$$\begin{aligned} \frac{\partial F_i}{\partial x}(z,x,y) &= \frac{\partial f}{\partial x}(z,x,y) + \rho_i \frac{\partial f_i^+}{\partial x}(z,x,y) + \epsilon \frac{\partial \rho_i}{\partial x} f_i^+(z,x,y) \\ &= \frac{\partial f}{\partial x}(z,x,y) - 2\rho_i x + \epsilon \frac{\partial \rho_i}{\partial x} f_i^+(z,x,y) \\ &< 0. \end{aligned}$$

where $\dfrac{\partial f}{\partial x}(z, x, y) < 0$ near C_i and ϵ is chosen to be sufficiently small to guarantee overall negative value.

Moreover, if $p \in C_i$ is a critical point of F_i, then its index $\mathrm{ind}_{F_i}(p) = \mathrm{ind}\,(C_i) + \mathrm{ind}_{f_i}(p)$ is the sum of the index of p as a critical point of f_i in C_i and the index of C_i. And the Hessian of F_i at p has the matrix form as

$$\mathrm{d}^2 F_i|_p = \mathrm{d}^2 f|_p + \mathrm{d}^2 f_i^+|_p = \begin{pmatrix} \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} & & \\ & + & \\ & & - \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} + & \\ & - \end{pmatrix} & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

Repeat this process to f_i on C_i for each i, we obtain a slightly perturbed Morse function F_ϵ out of the original f and f_i^+'s as

$$F_\epsilon = f + \epsilon \sum_i \rho_i f_i^+.$$

Denote v_ϵ and v_i to be gradient vector fields of F_ϵ and f_i respectively with v_ϵ close to v. Now we construct a chain complex homotopic to the simplicial chain complex according to our relative Morse theory of Chapter 1:

Definition 5.2.2 Let F_ϵ be a Morse approximation of a Morse-Bott function f on M, and v_ϵ be the gradient vector field of F_ϵ, then we define the Morse-Bott complex of f as a direct sum of Morse chain complexes of the critical manifolds, and the boundary maps are derived from gradient v_ϵ:

$$C_*^{\mathrm{MB}}(M, B, f) = C^{\mathrm{MS}}(M, B, F_\epsilon, v_\epsilon) = \bigoplus_i C^{\mathrm{MS}}_{*-\mathrm{ind}\,(C_i)}(C_i, f_i),$$

and the boundary map $\partial_\lambda : C_\lambda^{\mathrm{MB}}(M, B, f) \to C_{\lambda-1}(M, B, f)$ is defined exactly the same as in the Morse function case:

$$\partial_\lambda(p) = \sum_{q \in \mathrm{Crit}\,_{\lambda-1}(F_\epsilon)} [p : q]q,$$

where $[p : q]$ is the incidence coefficient by counting trajectories of $-v_\epsilon$ from p to q with given signs according to a prescribed orientation of $W^s(p, v_\epsilon)$ and $W^u(q, v_\epsilon)$.

Now we want to show that we are able to construct F_ϵ with care so that the gradient v_ϵ of F_ϵ preserves the ordering according to our algorithm in the beginning of the chapter.

The following lemma makes this possible:

Lemma 5.2.3 For each two connected components $C_i, C_j \subset \mathbf{C}$ of $\mathbf{C}$ such that

$$W^u(C_i, v) \cap W^s(C_j, v) = \emptyset,$$

and there are no broken trajectories between them, then there exist open neighbourhoods $N(C_i)$ and $N(C_j)$ so that

$$W^u(N(C_i), v) \cap W^s(N(C_j), v) = \emptyset.$$

In other words, the lemma states that if there are no trajectories flowing from C_i to C_j, then we can find some small neighbourhoods of C_i and C_j so that there are no trajectories flowing from the neighbourhood of C_i to the one of C_j. Note that, when $i < j$, according to our algorithm, it is automatically true that there exist no broken trajectories from C_i to C_j.

Proof: We prove it by contradiction. Firstly, let us assume $f(C_i) < f(C_j)$, otherwise, the gradient of f will guarantee no flow from the level set at C_i to the level set at C_j.

Suppose for the i, j such that $W^u(C_i, v) \cap W^s(C_j, v) = \emptyset$, any open neighbourhoods $N(C_i)$ and $N(C_j)$ there always exist trajectories from $N(C_i)$ to $N(C_j)$:

$$W^u(N(C_i), v) \cap W^s(N(C_j), v) \neq \emptyset.$$

Let $\{N_k(C_i)\}$ and $\{N_l(C_j)\}$ be sequences of nested neighbourhoods of C_i and C_j, respectively, with $N_{k+1}(C_i) \subset N_k(C_i)$ and $N_{l+1}(C_j) \subset N_l(C_j)$, so that

$$W^u(N_k(C_i), v_\epsilon) \cap W^s(N_l(C_j), v_\epsilon) \neq \emptyset,$$

and

$$\bigcap_k N_k(C_i) = C_i \text{ and } \bigcap_l N_l(C_j) = C_j.$$

Now for each $k = l$, choose a trajectory γ_k from $N_k(C_i)$ to $N_k(C_j)$. We want to show that such condition leads us to one of the two situations that contradict the hypothesis. Namely, the sequence $\{\gamma_k\}_k$ of trajectories between neighbourhoods of C_i and C_j will converge to either a trajectory from C_i to C_j or a broken trajectory that pass by some $C_{j'}$, where $C_{j'}$ is a critical manifold that lies between C_i and C_j, meaning $f(C_i) < f(C'_j) < f(C_j)$. (In fact, there can be more than one critical manifold between C_i and C_j, but the argument is essentially the same, so we stick to the simplest case. On the other hand, in the case of no such critical manifold in the middle, the argument can be simplified accordingly.)

Suppose $C_{j'}$ is such a connected critical manifold between C_i and C_j. Then let $\{x_k^1\}$ and $\{x_k^2\}$ be the two sequences of points lying on trajectories between the nested neighbourhoods $\{N_k(C_i)\}$ and $\{N_k(C_j)\}$. Moreover, we assume $\{x_k^1\}$ is between C_i and $C_{j'}$, and $\{x_k^2\}$ is between $C_{j'}$ and C_j. Since $M - \big(N_0(C_i) \cup N_0(C_j)\big)$ is compact, the limits x^1 and x^2 are contained in $M - \big(N_0(C_i) \cup N_0(C_j)\big)$, and we claim the trajectory where x^1 lies originates from C_i.

Suppose the trajectory in which x^1 lies does not come from C_i. Then we can assume that there exists some k_0 such that $\gamma_{x^1} \cap N_{k_0}(C_i) = \emptyset$. By the continuity of the flow, there exists a neighbourhood $N(x^1)$ of x^1 such that for any points $y \in N(x^1)$ the trajectory $\gamma_y \cap N_{k_0}(C_i) = \emptyset$. Since x^1 is the limit of a subsequence of $\{x_k^1\}$, there exist $x_{k_n} \in \{x_k\}$ such that $x_{k_n} \in N(x^1)$ also lie in $N(x^1)$ for each $k_n > k_0$. In particular, this says $x_{k_n} \notin W^u(N_{k_n}(C_i), v) \cap W^s(N(C_{j'}), v_\epsilon)$ for any

$N(C_{j'})$, i.e. $\gamma_{x_{n_k}}$ does not belong to $W^u(N_{n_k}(C_i), v)$, a contradiction to our choice of the collection $\{\gamma_k\}_k$. Therefore, x^1 comes from C_i. Symmetrically, x^2 ends in C_j.

Finally, consider the trajectory originates from C_i where x^1 lies, it can either ends in $C_{j'}$ first and then resumes there and reaches C_j after passing through x^2; or it bypasses $C_{j'}$ and reaches C_j directly. But either way, it contradicts to the hypothesis in the statement. □

Proposition 5.2.4 With the above notation, for each $C_\alpha \in \mathcal{A}(j)$, there exists an open neighbourhood $N(C_\alpha)$ of C_α so that the $N(C_\alpha)$ respects the ordering of the algorithm, in other words, replace C_α by $N(C_\alpha)$, the algorithm will label $N(C_\alpha)$ by j the same as C_α. □

Then choose v_ϵ sufficiently close to v, we obtain a filtration of $C_*^{\mathrm{MB}}(M, B, f)$ by writing

$$C_*^{(k)}(F_\epsilon) = \bigoplus_{j=1}^{k} \bigoplus_{\alpha \in \mathcal{A}(j)} C_*^{\mathrm{MS}}(C_\alpha, f_\alpha),$$

where $f_\alpha : C_\alpha \to \mathbb{R}$ is the Morse function that approximates f on C_α. Then

$$C_*^{(1)}(F_\epsilon) \hookrightarrow C_*^{(2)}(F_\epsilon) \hookrightarrow \cdots \hookrightarrow C_*^{(n)}(F_\epsilon) = C_*^{\mathrm{MB}}(M, B, f)$$

induces the desired spectral sequence with E^1 term:

$$E^1_{k,l} = \bigoplus_{\alpha \in \mathcal{A}(k)} H_{k+l-\mathrm{ind}\,(C_\alpha)}(C_\alpha, f_\alpha) \cong \bigoplus_{\alpha \in \mathcal{A}(k)} H_{k+l-\mathrm{ind}\,(C_\alpha)}(C_\alpha).$$

We summarise the result in the following corollary:

Corollary 5.2.5 The homology of the Morse-Bott complex is isomorphic to the relative homology of the underlying manifold, therefore, the spectral sequence induced by the filtered Morse-Bott complex converges to the relative homology:

$$E^r_{k,l} \Rightarrow H^{\mathrm{MB}}_{k+l}(M, B, f) \cong H_{k+l}(M, B) \text{ when } r \to \infty,$$

with E^1 term as:

$$E^1_{k,l} \cong \bigoplus_{\alpha \in \mathcal{A}(k)} H^{\mathrm{MS}}_{k+l-\mathrm{ind}\,(C_\alpha)}(C_\alpha, f_\alpha) \cong \bigoplus_{\alpha \in \mathcal{A}(k)} H_{k+l-\mathrm{ind}\,(C_\alpha)}(C_\alpha),$$

where $f_\alpha : C_\alpha \to \mathbb{R}$ is the Morse function that approximates f on C_α. □

Morse inequalities

Notation 5.2.6 Denote $\beta_n = \mathrm{rank}\,(H_n(M,B))$ and $\beta_n(C(k)) = \mathrm{rank}\,(H_n(C(k)) = \sum_{\alpha\in\mathcal{A}(k)} \dim H_n(C_\alpha)$.

Corollary 5.2.7 There is a polynomial $R(t)$ with non-negative coefficients such that

$$\sum_{k,l} \beta_{k+l-\mathrm{ind}\,(C(k))}(C(k))t^{k+l} = \sum_n \beta_n t^n + (1+t)R(t)$$

Proof: From Theorem 5.2.1 we know:

$$\beta_n = \sum_{k+l=n} \mathrm{rank}\,(E^\infty_{k,l})$$

We want to show that there exists non-negative polynomial $R(t)$ such that:

$$\sum_n \sum_{k+l=n} t^n \mathrm{rank}\,(E^1_{k,l}) = \sum_n \sum_{k+l=n} t^n \mathrm{rank}\,(E^\infty_{k,l}) + (1+t)R(t).$$

Since the filtration is finite and the spectral sequence will reach stability after finite pages, and it can be reduced to show for each r,

$$\sum_n \sum_{k+l=n} t^n \mathrm{rank}\,(E^r_{k,l}) = \sum_n \sum_{k+l=n} t^n \mathrm{rank}\,(E^{r+1}_{k,l}) + (1+t)R_r(t),$$

where $R_r(t)$ is a non-negative polynomial.

Denote $Z^r_{k,l} = \ker(d_r : E^r_{k,l} \to E^r_{k-r,l+r-1})$ and similarly, $B^r_{k,l} = \mathrm{Im}\,(d_r : E^r_{k,l} \to E^r_{k-r,l+r-1})$, then

$$\mathrm{rank}\,(E^r_{k,l}) = \mathrm{rank}\,(Z^r_{k,l}) + \mathrm{rank}\,(B^r_{k,l}), \tag{5.1}$$

and by the construction of the spectral sequence, $E^{r+1}_{kl} = Z^r_{kl}/B^r_{k+r,l-r+1}$, therefore,

$$\mathrm{rank}\,(E^{r+1}_{k,l}) = \mathrm{rank}\,(Z^r_{k,l}) - \mathrm{rank}\,(B^r_{k+r,l-r+1}), \tag{5.2}$$

so

$$\mathrm{rank}\,(E^r_{k,l}) = \mathrm{rank}\,(E^{r+1}_{k,l}) + \mathrm{rank}\,(B^r_{k,l}) + \mathrm{rank}\,(B^r_{k+r,l-r+1}),$$

therefore

$$\sum_{k+l=n} \mathrm{rank}\,(E^r_{k,l}) = \sum_{k+l=n} \mathrm{rank}\,(E^{r+1}_{k,l}) + \sum_{k+l=n} \mathrm{rank}\,(B^r_{k,l}) + \sum_{k+l=n} \mathrm{rank}\,(B^r_{k+r,l-r+1}) \tag{5.3}$$

Now take the alternating sum of the above Equation (5.3) from $n=0$ to $n=N$ for any N, we have

$$\sum_{n=0}^{n=N}\sum_{k+l=n}(-1)^{N-n}\mathrm{rank}\,(E^r_{k,l}) = \sum_{n=0}^{n=N}\sum_{k+l=n}(-1)^{N-n}\mathrm{rank}\,(E^{r+1}_{k.l}) + \sum_{p+q=k}\mathrm{rank}\,(B^r_{p,q}).$$

This is equivalent to

$$\sum_{n=0}^{\infty}\sum_{k+l=n}t^n\mathrm{rank}\,(E^r_{k,l}) = \sum_{n=0}^{\infty}\sum_{k+l=n}t^n\mathrm{rank}\,(E^{r+1}_{k,l}) + (1+t)R_r(t),$$

for some non-negative polynomial $R_r(t)$.

Suppose the spectral sequence becomes stable at page r_0, i.e., $E^{r_0} = E^\infty$, then after repeating this process r_0 times, and taking the sum of them all, we have the following:

$$\sum_{n}\sum_{k+l=n}t^n\mathrm{rank}\,(E^1_{k,l}) = \sum_{n}\sum_{k+l=n}t^n\mathrm{rank}\,(E^{r_0}_{k,l}) + (1+t)\sum_{r=1}^{r_0-1}R_r(t).$$

Now $R(t) = \sum_{r=1}^{n-1}R_r(t)$ is non-negative and the proof is complete. □

5.2.2 For circle-valued functions

For general closed 1-forms, we follow the previous treatment as in the Morse situation, where we approximate such a 1-form to a particular type, a rational closed 1-form, which in turn is equivalent to the study of the circle-valued functions on a manifold with boundary.

The settings

Let us recall the essential settings laid out in Chapter 2:

For a compact manifold M with boundary ∂M, let $f : M \to S^1$ be a circle-valued function on M with exit set B, which is defined as in Chapter 2. Denote $\mathbf{C}$ as the critical set of f on M, since locally f is seen to be real, we call f satisfying the *Bott nondegeneracy* condition, if restrict to a tubular neighbourhood of each connected component C_i of $\mathbf{C}$, f satisfies conditions **B1** and **B2**.

Assuming $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ is surjective. For simplicity, we consider the minimal covering space $\bar\rho : \bar M \to M$ whose fundamental group $\pi_1(\bar M)$ corresponds to the kernel of f_*, i.e. the covering transformation group of $\bar{\rho}$ is the infinite cyclic group $\mathbb{Z}$. We are aware the rational approximation and the choice of minimal covering would lose considerable amount of information about the closed 1-form in terms of its cohomology class and the Novikov ring, nevertheless, the essential geometric picture will be retained.

We can find a lift $\bar{f} : \bar{M} \to \mathbb{R}$ of $f\bar{\rho}$, and denote $M^1_\mu = \bar{f}^{-1}((-\mu, 1])$, $M_\mu = \bar{f}^{-1}(\{-\mu\})$ and $B_\mu = \bar{B} \cap M^1_\mu$ for integer $\mu \geq 0$. Furthermore, if v is a gradient vector field of f with respect to some Riemannian metric, then denote its lift on the covering space $\bar{v}$.

Suppose we have the Novikov-Sikorav ring as

$$\mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}] = \{\lambda = \Sigma^{\infty}_{-\infty} a_\mu t^\mu : |\{t^\mu \text{ with } a_\mu \neq 0, \mu \leq 0\}| < \infty\},$$

We want to construct a chain complex based on the critical manifold over the Novikov-Sikorav ring, and understand its homology.

Firstly, let us make the condition of no homoclinic cycle precise:

Definition 5.2.8 Let $\{C_i\}$ be the set of connected components of the Bott's critical manifold $\mathbf{C}$ in M, a homoclinic cycle of length n is a finite sequence of trajectories $\gamma_1, \ldots, \gamma_n, \gamma_{n+1} = \gamma_1$ in M satisfies:

$$\lim_{t \to +\infty} \gamma_j \cup \lim_{t \to -\infty} \gamma_{j+1} \subset C_{i(j)}$$

for all $1 \leq j \leq n$.

The concept of homoclinic cycle will be further explored in the next chapter, where we deal with its relationship with the category $\mathrm{cat}(X, B, [\omega])$ with respect to a closed 1-form ω.

No Homoclinic Cycle Condition

Let ω be a closed 1-form on M nondegenerate in the sense of Bott, we assume that there exists a gradient vector field v of ω for a given Riemannian, whose flow contains no homoclinic cycles. $\square$

With the no homoclinic cycle assumption, we are able to perform the algorithm describe in the beginning of the chapter and get an ordering $\{j\}$ as for each connected component C_i of $\mathbf{C}$ so that after reindexing, each C_α has order j for $\alpha \in \mathcal{A}(j)$. However, the flow of gradient v in M can still circulate around the whole M which makes the filtration in the base space impossible. Instead, we look at each piece M^1_μ of the covering space, then we can have similar filtration as in the real function case. Denote ${}_\mu C(j)$ as the union of connected components with order j in M^1_μ, we make the following simple observation:

Observation: Assuming the no homoclinic condition, the ordering of the critical manifold $\mathbf{C}$ is preserved in the covering space, in other words,

$$ {}_{\mu_0}C(j) = \bigcup_{\mu=0}^{\mu_0} t^\mu \bar{C}(j), $$

where each $\bar{C}(j) \subset M^1_0$ is a chosen lift of the union of connected component C_α for $\alpha \in \mathcal{A}(j)$ in M^1_0 and $t \in \mathbb{Z} = \langle t \rangle$ is a generator with $f_*(t) = -1$.

Filtration of the Novikov-Bott complex

For each μ we want to construct the filtration as in the real function case. The new obstruction here to overcome is that each time when we approximate $\bar{f}$ on a finite pieces M^1_μ by F^μ, we get a slightly different gradient vector field v^μ. This increases the complexity of getting chain maps for the inverse system, and we want to concentrate on this aspect of the construction.

Note also that the approach here is roughly analogous to Chapter 2, however, the new obstacle we are facing here is that the Bott condition does not directly imply self-indexing property on the critical set. So in order to preserve the ordering and no homoclinic cycle condition as in the real function case, we have to adjust the modification F^μ of $\bar{f}$ for each μ.

Consider the base manifold M, according to Lemma 5.2.3, for each connected component C_i, there exists a neighbourhood $N_\mu(C_i)$ for each μ so that on the total space $\bar{M}$, in the first μ copies of M^1_0 we have $\overline{N_\mu(C_i)}$ respecting the no homoclinic cycle assumption as well as C_i for each i. With these neighbourhoods in hand for each C_i, we can approximate f by some Morse function F^μ so that the lift of the gradient v^μ of F^μ on the covering observes the no homoclinic cycle assumption for the first μ copies, and therefore respects the ordering. With slight abuse of notation, we denote the lifts of F^μ and v^μ restricting to M^1_μ with the same notations F^μ and v^μ. Let $C^{\mathrm{MS}}_*(F^\mu, v^\mu)$ be the relative Morse complex of F^μ on M^1_μ, then we can define the Morse-Bott complex for $(M^1_\mu, B_\mu \cup M_\mu)$ as in the previous section:

$$\begin{aligned} C^{\mathrm{MB}}_*(M^1_\mu, B_\mu \cup M_\mu, \bar{f}|_{M^1_\mu}, v^\mu) &= \bigoplus_{k=0}^{\mu} \bigoplus_i C^{\mathrm{MS}}_{*-\mathrm{ind}\,(C_i)}(t^k C_i, f_i) \\ &= \bigoplus_{j=1}^{n} C_{*-\mathrm{ind}\,({}_\mu C(j))}({}_\mu C(j), f_j), \end{aligned}$$

for each Morse function $f_j : {}_\mu C(j) \to \mathbb{R}$ independent of μ. Then by denoting

$${}_\mu C^{(k)}(F_\mu) = \bigoplus_{j=1}^{k} C_{*-\mathrm{ind}\,({}_\mu C(j))}({}_\mu C(j), f_j)$$

we have the following filtration:

$${}_\mu C^{(0)}(F_\mu) \hookrightarrow {}_\mu C^{(1)}(F_\mu) \hookrightarrow \cdots \hookrightarrow {}_\mu C^{(n)}(F_\mu) = C^{\mathrm{MB}}_*(M^1_\mu, B_\mu \cup M_\mu, \bar{f}|_{M^1_\mu}, v^\mu)$$

Notice that when μ gets bigger each time, the tubular neighbourhood of each connected critical component can be chosen to be smaller for the Morse approximation F^μ. Moreover, for each μ, $C^{\mathrm{MB}}_*(M^1_\mu, B_\mu \cup M_\mu, \bar{f}|_{M^1_\mu}, v^\mu)$ can be seen as a free $\mathbb{Z}[t]/t^\mu$ module generated by critical points of F^μ, where t is a generator of the covering transformation group $\mathbb{Z}$.

Before the actual construction, we like to shorten the key notations to the following:

Notation 5.2.9 Let $C^\Delta(\mu) = C^\Delta_*(M^1_\mu, B_\mu \cup M_\mu)$ and $C^{\mathrm{MB}}(\mu) = C^{\mathrm{MB}}_*(M^1_\mu, B_\mu \cup M_\mu, \bar{f}|_{M^1_\mu}, v^\mu)$.

In order to obtain a inverse system where each map $C^{\mathrm{MB}}_*(\mu+1) \to C^{\mathrm{MB}}_*(\mu)$ is an honest projection of $\mathbb{Z}[t]/t^{\mu+1}$ module to $\mathbb{Z}[t]/t^\mu$ module, we want to choose these neighbourhoods carefully so that $v^{\mu+1}$ is very close to v^μ and the chain map $C^{\mathrm{MB}}_*(\mu, v^{\mu+1}) \to C^{\mathrm{MB}}_*(\mu)$ is an identity, where $C^{\mathrm{MB}}_*(\mu, v^{\mu+1})$ is the Morse-Bott complex of M^1_μ with gradient $v^{\mu+1}$ instead of v^μ. More general, we want:

$$C^{\mathrm{MB}}_*(\mu, v^{\mu'}) = C^{\mathrm{MB}}_*(\mu), \text{ for any } \mu' \geq \mu.$$

We do this inductively. Suppose we already have the neighbourhoods and the gradient vector field v^μ for M^1_μ. On $M^1_{\mu+1}$, we can choose each neighbourhood $N_{\mu+1}(C_i) \subset N_\mu(C_i)$ smaller and this guarantees the same ordering of each neighbourhood $N_{\mu+1}(C_i)$ following the algorithm in Section 5.2.1. Recall the Definition 1.3.13 and notice $v^{\mu+1}$ and v^μ coincide on the complement of the union of all the neighbourhoods $N_\mu(C_i)$, and share the same critical points on M^1_μ. Therefore, for any $p \in M^1_\mu$, $W^u(p, v^{\mu+1}) \cap W^s(q, v^\mu) = \emptyset$ where $p \in \operatorname{Crit} F^\mu$ and $p \neq q$ with $\operatorname{ind} q = \operatorname{ind} p$. In other words, the incidence number $[p : q]$ between two distinct critical points p, q is trivial, when $v^{\mu+1}$ is chosen sufficiently close to v^μ . With this construction in hand, we readily get the following commutative diagram where the horizontal maps $\mathrm{Pr}^{\mathrm{MB}}$ are indeed honest projections:

$$\begin{tikzcd}
\cdots \arrow[r] & C_*^{\mathrm{MB}}(\mu+1) \arrow[rr, "\mathrm{Pr}^{\mathrm{MB}}"] \arrow[dr] & & C_*^{\mathrm{MB}}(\mu) \arrow[r] & \cdots \\
& & C_*^{\mathrm{MB}}(\mu, v^{\mu+1}) \arrow[ur, "="'] & &
\end{tikzcd} \tag{5.4}$$

Now we want to have the following commutative diagram to compare the simplicial chain complex and the Morse-Bott complex for each μ, and hence understand the homology of the inverse limit:

$$\begin{tikzcd}
C_*^{\Delta}(\mu+1) \arrow[r, "\mathrm{Pr}^{\Delta}"] \arrow[d, "\varphi_{\mu+1}"] & C_*^{\Delta}(\mu) \arrow[d, "\varphi_\mu"] \\
C_*^{\mathrm{MB}}(\mu+1) \arrow[r, "\mathrm{Pr}^{\mathrm{MB}}"] & C_*^{\mathrm{MB}}(\mu)
\end{tikzcd} \tag{5.5}$$

where $\mathrm{Pr}^{\mathrm{MB}}, \mathrm{Pr}^{\Delta}$ are projections of the Morse complexes and simplicial chain complexes respectively, and φ_μ is a chain map for each μ.

We summarise our goal in the following lemma:

Lemma 5.2.10 There exists a sequence $\{v^\mu\}_{\mu=0,\ldots}$ of vector fields v^μ for each μ on the covering $\bar{M}$ so that we have the above commutative Diagram (5.5).

Proof: We do the construction inductively.

To define a chain map between each $C_*^{\Delta}(\mu)$ and $C_*^{\mathrm{MB}}(\mu)$, let $\Psi_\mu : M \to M$ be a diffeomorphism isotopic to $\Psi_{\mu-1}$. Viewing a triangulation Δ as a diffeomorphism from a simplicial complex onto M as in [31, §8.3], then we want the composition $\Psi_\mu\Delta$ to be a triangulation adjusted to v^μ for each μ. Suppose again this is done inductively up to μ pieces on $\bar{M}$. This is possible when v^μ is chosen close to $v^{\mu-1}$ (which we did) and the construction of Ψ_μ follows from [29, Lemma 5.3], therefore we have the chain map $\varphi_\mu : C_*^\Delta(\mu) \to C_*^{\mathrm{MB}}(\mu)$ for each μ.

Now suppose we have built the chain map $\varphi_\mu : C_*^\Delta(\mu) \to C_*^{\mathrm{MB}}(\mu)$ for μ by choosing a triangulation adjusted to v^μ. Notice that v^μ is also a gradient of $F^{\mu+1}$ on $M^1_{\mu+1}$ for a suitable Riemannian metric, as $v^\mu(F^{\mu+1}) > 0$ for the points away from the critical set in $M^1_{\mu+1}$. Nevertheless, we still need the transversality to make a chain complex, so let $v^{\mu+1}$ be chosen based on the following three criteria:

1. Gradient $v^{\mu+1}$ of $F^{\mu+1}$ is Smale-transverse on $M^1_{\mu+1}$;
2. The chain map $\mathrm{Pr}^{\mathrm{MB}} : C_*^{\mathrm{MB}}(\mu+1) \to C_*^{\mathrm{MB}}(\mu)$ is a projection.
3. The intersection $W^u(p, v^{\mu+1}) \cap \sigma$ is transverse on M^1_μ for each $\sigma \in \Psi_\mu(\Delta)$. Moreover, the intersection number is the same as $W^u(p, v^\mu) \cap \sigma$ on M^1_μ.

Note that Criterion 1 is standard, and we have achieved Criterion 2 following the construction that leads to Diagram (5.5). We will elaborate Criterion 3 in the following. So far, we have built Diagram (5.5), and the rest of the proof is dedicated to its commutativity.

Consider an i-simplex $\sigma \in C_i^\Delta(\mu+1)$, there are two cases: $\sigma \subset M^1_{\mu+1} \backslash M^1_\mu$ and $\sigma \subset M^1_\mu$. For the first case,

$$\mathrm{Pr}^{\mathrm{MB}}\varphi_{\mu+1}(\sigma) = \mathrm{Pr}^{\mathrm{MB}}\Big(\sum_{p\in \mathrm{Crit}\,_i F^{\mu+1}} [\sigma : p]_{\mu+1} p\Big) = 0;$$

where $[\cdot : \cdot]_{\mu+1}$ is the incidence number under the vector field $v^{\mu+1}$, and it becomes 0 after the projection because all the nontrivial incidence numbers only arise when the critical points are in $M^1_{\mu+1} \backslash M^1_\mu$. And similarly $\mathrm{Pr}^\Delta(\sigma) = 0$.

For the second case,

$$\mathrm{Pr}^{\mathrm{MB}}\varphi_{\mu+1}(\sigma) = \mathrm{Pr}^{\mathrm{MB}}\Big(\sum_{p\in \mathrm{Crit}\,_i F^{\mu+1}} [\sigma : p]_{\mu+1} p\Big) = \sum_{p\in \mathrm{Crit}\,_i F^{\mu}} [\sigma : p]_{\mu+1} p,$$

whereas

$$\varphi_\mu \mathrm{Pr}^\Delta(\sigma) = \varphi_\mu(\sigma) = \sum_{p\in \mathrm{Crit}\,_i F^{\mu}} [\sigma : p]_{\mu} p.$$

Now we reduce the commutativity problem to the following equality:

$$[\sigma : p]_\mu = [\sigma : p]_{\mu+1}.$$

Our aim now is to describe our choice of $v^{\mu+1}$ and $\Psi_{\mu+1}$ so that $\Psi_{\mu+1}(\sigma) \cap W^u(p, v^{\mu+1})$ has the same number of intersection points as $\Psi_\mu(\sigma) \cap W^u(p, v^\mu)$.

When the incidence number is trivial, i.e. no intersection, then we can always find $v_{\mu+1}$ sufficiently close to v_μ so that there is no intersection of the simplex with the unstable manifold.

Otherwise, notice there are only finite number of intersection points of $\sigma \cap W^u(p, v^\mu)$ by compactness of σ, without loss of generality, and we assume there is only one intersection point z in the interior of σ.

Firstly, we make our choice of $v^{\mu+1}$ precise with respect to Criterion 3 above. According to [5, Theorem 7.7], let $\operatorname{ind} p = k$, then we can choose a neighbourhood U of z with some suitable local coordinates, so that we have the following parametrisation:

$$U \cap W^u(p, v^\mu) \to \{0\} \times \mathbb{R}^{m-k}, \tag{5.6}$$

and

$$U \cap \sigma \to \mathbb{R}^k \times \{0\}.$$

In particular, $z \to (0,0)$ for $z \in \sigma \cap W^u(p, v^\mu)$.

Let $i : \mathbb{R}^{m-k} \to M$ be the injective immersion of the unstable manifold $W^u(p, v^\mu)$ so that the composition with map (5.7) maps $0 \to z \to (0,0)$. Choose unit disk $D \subset \mathbb{R}^{m-k}$ centred at 0, and modify the parametrisation $U \cap W^u(p, v^\mu) \to \{0\} \times \mathbb{R}^{m-k}$ if necessary so that the composition

$$I : D \xrightarrow{i_D} U \cap W^u(p, v^\mu) \to 0 \times \mathbb{R}^{m-k}$$

is the identity map mapping $x \to (0, x)$.

Now according to [3, Parametrisation Theorem 27.4], we can choose $v^{\mu+1}$ arbitrarily C^1 close to v^μ so that the injective immersion $i : \mathbb{R}^{m-k} \to M$ of $W^u(p, v^{\mu+1}$ is C^1 close to i. In particular, with respect to the parametrisation of U, the following composition $i' : D \xrightarrow{j_D} U \cap W^u(p, v^{\mu+1}) \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ is defined as

$x \to (g(x), h(x)) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ for $x \in D$ so that $g(x)$ and $h(x)$ are C^1 close to 0 and $f(x)$ respectively. To be precise, let

$$||\mathrm{d}h - \mathrm{d}I|| \leq \frac{1}{2},$$

then the procedure in the proof of Inverse Function Theorem, for instance [24, pp 160], shows that h is injective in D. By choosing also that $||h(x) - x|| \leq \frac{1}{2}$ for all $x \in D$, then it is guaranteed that 0 is contained in the interior of $h(\partial D)$.

Now we manage to show the uniqueness of the intersection, in order to show the existence, i.e. the solution of $h(x) = 0$ exists, we devise the following argument:

According to the generalised Jordan Curve Theorem, h maps the boundary ∂D into $\mathbb{R}^{m-k}$ so that the interior of $h(\partial D)$ is a $m-k$-dimensional disk D'. Collapse the boundary $h(\partial D)$ of D', we have an $m-k-1$ sphere S', similarly we get $m-k-1$ sphere S from D, so that we have map $h' : S \to S'$, now h' is smooth and injective at least away from the collapsing point, and the degree of h' is 1, which shows which h' is surjective and hence h, i.e. the solution of $h(x) = 0$ exists.

With the similar process, we can find a diffeomorphism $\Psi_{\mu+1}$ isotopic to Ψ_μ so that $\Psi_{\mu+1}(\sigma)$ intersects $W^u(p, v^{\mu+1})$ transversely on $M^1_{\mu+1}$ at the same number of points.

We illustrate the idea in the following Figure 5.1:

Therefore, by our inductive choice of v^μ and Ψ_μ, we show the diagram is commutative. $\square$

Taking the inverse limit of the system (5.5) and tensoring it with the Novikov ring $\mathbb{Z}((t))$, now we finish the construction of our Novikov-Bott complex:

Definition 5.2.11 We define the *Novikov-Bott* complex as

$$C^{\mathrm{NovB}}_*(M, B, f) = \mathbb{Z}((t)) \otimes \varprojlim_{\mu} C^{\mathrm{MB}}_*(\mu).$$

Proposition 5.2.12 For each μ, we have chain homotopy equivalence $\varphi_\mu : C^\Delta_*(\mu) \simeq C^{\mathrm{MB}}_*(\mu)$. $\square$

This is the result of Chapter 1. We now want to understand the homology of the inverse limit of the above system tensored with the Novikov ring $\mathbb{Z}((t))$:

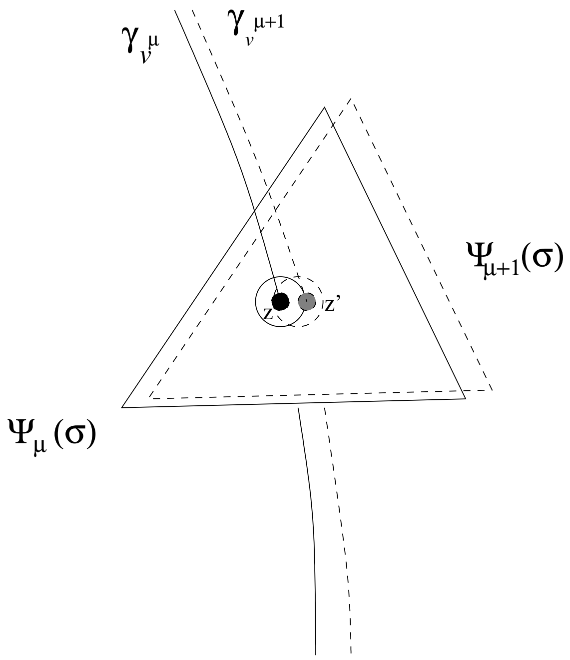

Figure 5.1: Choice of $v^{\mu+1}$ and $\Psi_{\mu+1}$

Since (5.5) satisfies the Mittag-Leffler condition, the inverse system (5.5) induces the following exact sequence:

$$0 \to \varprojlim{}^1 H_{i+1}(C^{\mathrm{MB}}(\mu)) \to H_i(\varprojlim C^{\mathrm{MB}}(\mu)) \to \varprojlim H_i(C^{\mathrm{MS}}(\mu)) \to 0,$$

and similarly,

$$0 \to \varprojlim{}^1 H_{i+1}(C^{\Delta}(\mu)) \to H_i(\varprojlim C^{\Delta}(\mu)) \to \varprojlim H_i(C^{\Delta}(\mu)) \to 0.$$

Therefore applying the Five Lemma to the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim{}^1 H_{i+1}(C^{\Delta}(\mu)) \arrow[r] \arrow[d, "\cong"] & H_i(\varprojlim C^{\Delta}(\mu)) \arrow[r] \arrow[d] & \varprojlim H_i(C^{\Delta}(\mu)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \varprojlim{}^1 H_{i+1}(C^{\mathrm{MB}}(\mu)) \arrow[r] & H_i(\varprojlim C^{\mathrm{MB}}(\mu)) \arrow[r] & \varprojlim H_i(C^{\mathrm{MB}}(\mu)) \arrow[r] & 0.
\end{tikzcd}$$

we have shown the isomorphism in the middle term $H_i(\varprojlim C^{\Delta}(\mu)) \to H_i(\varprojlim C^{\mathrm{MB}}(\mu))$ for each i.

Now since both chain complexes are free, the isomorphism of the homology groups induces the chain homotopy equivalence of the following two chain complexes after tensoring with the Novikov ring $\mathbb{Z}((t))$:

$$\mathbb{Z}((t)) \otimes \varprojlim_{\mu} C_*^{\mathrm{MB}}(\mu) \simeq \mathbb{Z}((t)) \otimes C^{\Delta}(\bar{M}, \bar{B}), \tag{5.7}$$

and it also respects the filtration, i.e. denote

$$\widehat{C}^{(k)} = \varprojlim_{\mu} {}_{\mu}C^{(k)}(\mu) = \varprojlim_{\mu} \bigoplus_{j=1}^{k} C_{*-\mathrm{ind}\,({}_{\mu}C(j))}({}_{\mu}C(j), f_j),$$

we have

$$\mathbb{Z}((t)) \otimes \widehat{C}^{(0)} \hookrightarrow \mathbb{Z}((t)) \otimes \widehat{C}^{(1)} \hookrightarrow \cdots \hookrightarrow \mathbb{Z}((t)) \otimes \widehat{C}^{(n)} = C^{\mathrm{NovB}}(M, B, f).$$

We summarise this section in the following statement:

Proposition 5.2.13 The filtered chain complex $C_*^{\mathrm{NovB}}(M, B, f, v)$ induces a spectral sequence with

$$E^1_{k,l} = \bigoplus_{\alpha \in \mathcal{A}(k)} H_{k+l-\mathrm{ind}\,(\bar{C}_\alpha)}(\bar{C}_\alpha; \mathbb{Z}((t))),$$
$$E^r_{k,l} \Rightarrow H_{k+l}\left(M, B; \mathbb{Z}((t))\right).$$

Here $\bar{C}_\alpha$ is a chosen lift of C_α in M^1_0 of the covering space with $\alpha \in \mathcal{A}(k)$ the index of the collection of order k connected components. □

Morse inequalities of a rational Novikov-Bott closed 1-form

Since the filtration is finite, E^*_{**} converges to the homology groups of M after finite pages, and we have

$$E^\infty_{k,l} = \frac{\mathrm{Im}\left(H_{k+l-\mathrm{ind}\,(C^{(k)})}(\widehat{C}^{(k)}, \mathbb{Z}((t))) \to H_{k+l}(M, \mathbb{Z}((t))\right)}{\mathrm{Im}\left(H_{k+l+1-ind(C^{(k-1)})}(\widehat{C}^{(k-1)}, \mathbb{Z}((t))) \to H_{k+l}(M, \mathbb{Z}((t)))\right)}$$

Meanwhile, according to Lemma 1.10 in [13], the Novikov ring $\mathbb{Z}((t))$ is a principle ideal domain, and the homology over such ring splits as a direct sum of a free module and cyclic modules by the structure theorem for finitely generated modules, see pp 9 in [41] for example. Therefore, this spectral sequence resolves the extension problem as well as in the real function case:

$$H_n(M, B; \mathbb{Z}((t))) = \bigoplus_{k+l=n} E^\infty_{k,l}$$

so we can recover the rank of $H_n(M, B; \mathbb{Z}((t)))$:

$$\mathrm{rank}\, H_n(M, B; \mathbb{Z}((t))) = \sum_{k+l=n} \mathrm{rank}\,(E^\infty_{k,l}).$$

Notation 5.2.14 Denote

$$\beta_n(\xi) = \operatorname{rank} H_n(M, B; \mathbb{Z}((t)))$$

and

$$\beta_n(C(k), \xi) = \operatorname{rank} H_n(C(k); \mathbb{Z}((t))).$$

This implies the Morse inequalities for a Novikov-Bott closed 1-form:

Lemma 5.2.15 There is a polynomial $R(t)$ with non-negative coefficients such that

$$\sum_{k,l} \beta_{l+k}(C(k), \xi) t^{k+l} = \sum_n \beta_n(\xi) t^n + (1+t)R(t).$$

$\square$

5.2.3 Examples of Novikov-Bott closed 1-forms with the No Homoclinic Cycle Condition

In the subsection, we construct a generic example that satisfies the No Homoclinic Cycle condition we took for granted since the beginning, and show it's actually preserved under Cartesian product.

Let M, N be compact manifolds with boundary. Let $f : M \to \mathbb{R}$ be a real function nondegenerate in the sense of Bott, and denote its critical set $S = \bigcup_i S_i$ as disjoint union of connected critical manifolds of f. Let ω be a Morse closed 1-form on N and denote its critical set $C = \{p_j\}$ as a set of Morse critical points of ω. Suppose we also fix two Riemannian metrics g_M and g_N on M and N respectively, then we have gradient vector fields u and v of f and ω with respect to g_M and g_N respectively.

Now consider the 1-form $\mathrm{d}f + \omega$. It is closed and has critical set $\bigcup_{i,j}(S_i, p_j)$. Take the product Riemannian metric (g_M, g_N) on $M \times N$, then $(u, v) : M \times N \to T(M \times N)$ is the gradient vector field of $\mathrm{d}f + \omega$ with respect to (g_M, g_N). For any two connected critical manifolds (S_i, p_j) and (S_k, p_l), it is easy to see that from (S_i, p_j) to (S_k, p_l) there exists no trajectory of the negative gradient $(-u, -v)$, if either $W^s(S_i, u) \cap W^u(S_k, u) = \emptyset$ or $W^s(p_j, v) \cap W^u(p_l, v) = \emptyset$. Because under both gradient fields u and v, homoclinic cycle is absent, the observation leads to the conclusion that $(-u, -v)$ satisfies the *no homoclinic cycle condition* on $M \times N$.

In fact, we can generalise the above construction to the Cartesian product of any number of manifolds with a Novikov-Bott closed 1-form:

Proposition 5.2.16 Let ω_i be Novikov-Bott closed 1-form on manifold M_i, for each $i = 1, \ldots, n$ where n is finite. Then with the product Riemannian metric, the gradient vector field of $\sum_i \omega_i$ satisfies the no homoclinic cycle condition on $\prod_i M_i$ if for each i, the gradient of ω_i satisfies the no homoclinic cycle condition on M_i. $\square$

Chapter 6

Lusternik-Schnirelman category with respect to a closed 1-form

In previous chapters, a substantial part is concerned with the estimation of the number of critical points of a closed 1-form. But such treatments require the Morse nondegeneracy condition of the critical points. In this chapter, we want to remove this restriction and describe a different approach to the question, a generalisation of the classical Lusternik-Schnirelman category initiated by M. Farber in [9] [10] [11] and [12]. In particular, we give a relative version of the concept and show the link to the absolute one via an inequality, which is achieved by a technical operation of segmenting the larger movable set, see Theorem 6.2.1. To make the exposition complete, we also provide analogous results on cuplength lower bound and estimate of critical numbers of a closed 1-form.

6.1 Definitions

We want to work on the more general topological spaces, so firstly let us define a continuous version of closed 1-form for topological spaces:

Definition 6.1.1 Let X be a topological space, *a continuous closed 1-form ω on X* is defined to be a collection $\{f_U\}_{U\in\mathscr{U}}$ of continuous real functions $f_U : U \to \mathbb{R}$, where

$\mathscr{U} = \{U\}$ is an open cover of X such that for any pair $U, V \in \mathscr{U}$, the difference

$$f_U|_{U\cap V} - f_V|_{U\cap V} : U \cap V \to \mathbb{R}$$

is locally constant.

Two continuous closed 1-forms $\omega_1 = \{f_U\}_{U\in\mathscr{U}}, \omega_2 = \{g_V\}_{V\in\mathscr{V}}$ are called *equivalent* if the union $\{f_U, g_V\}_{U\in\mathscr{U}, V\in\mathscr{V}}$ of the collections is a continuous closed 1-form, i.e. for any $U \in \mathscr{U}$ and $V \in \mathscr{V}$, the difference $f_U - g_V$ of the two functions f_U, g_V is locally constant on $U \cap V$. A trivial example for such topological continuous closed 1-forms can be constructed as follows:

Example 6.1.2 Suppose we take the whole space $\{X\}$ as the open cover, then any continuous function $f : X \to \mathbb{R}$ defines a continuous closed 1-form on X, denoted as $\mathrm{d}f$. It is the continuous version of the exact forms, we call it a *continuous exact form.*

In such an example, the two trivial closed 1-forms $\mathrm{d}f$ and $\mathrm{d}g$ are equivalent $\mathrm{d}f = \mathrm{d}g$ if and only if $f - g : X \to \mathbb{R}$ is locally constant, i.e. constant on each connected component of X.

The continuous version of the angular form can be constructed as follows:

Example 6.1.3 Consider the 1-dimensional sphere S^1, suppose it is parametrized by $t \to e^{\pi i t}$ and let it be covered by U, V where $U = (-\frac{1}{6}, \frac{7}{6})$ and $V = (\frac{5}{6}, \frac{13}{6})$ and the functions θ_U and θ_V are angular functions, i.e. $\theta_U(\alpha) = \pi\alpha$ for $\alpha \in U$ and $\theta_V(\beta) = \pi\beta$ for $\beta \in V$, then $\theta_V|_{U\cap V} - \theta_U|_{U\cap V} = 0, 2\pi$ locally constant, so $\mathrm{d}\theta = \{\theta_U, \theta_V\}$ is a continuous closed 1-form on S^1. It is easy to see that $\mathrm{d}\theta$ is not exact.

We want to define the integral of topological closed 1-forms, and it leads to their corresponding cohomology classes.

Definition 6.1.4 Suppose we have a closed 1-form $\omega = \{f_U\}_{U\in\mathscr{U}}$ for some open cover $\mathscr{U} = \{U\}$ of a topological space X, and $\gamma : [0, 1] \to X$ is a continuous path on X. The line integral $\int_\gamma \omega$ is defined as follows:

$$\int_\gamma \omega = \sum_{i=0}^{n-1} \left(f_{U_i}(\gamma(t_{i+1})) - f_{U_i}(\gamma(t_i))\right),$$

where $t_0 = 1 < t_1 < \cdots < t_n = 1$ is a partition of closed interval $[0,1]$ such that $\gamma[t_i, t_{i+1}] \subset U_i$ for all $1 \leq i \leq n$.

Remark 6.1.5 This integration is independent of the choice of partitions and the open cover $\mathscr{U}$, see Section 10.2 in [13].

The following standard results from [13, Chapter 10] lead to the cohomology class of a continuous closed 1-form.

Lemma 6.1.6 Suppose we have two continuous paths $\gamma_1, \gamma_2 : [0,1] \to X$ sharing the same end points, i.e. $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Moreover, if γ_1 and γ_2 are homotopic relative to the end points, then for any continuous closed 1-form ω on X, we have the equality:

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

We sketch a proof here:

Proof:

The idea of the proof can be shown in the following picture:

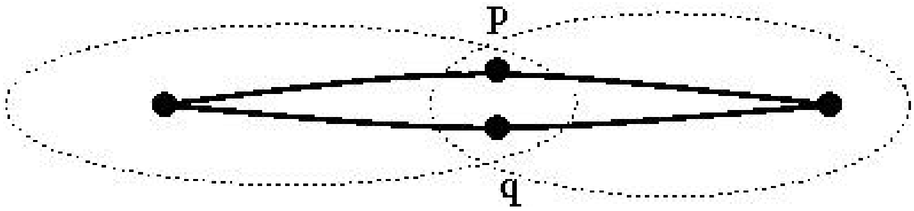

Figure 6.1: Homotopy Invariance

Here, without loss of generality, we can assume both paths are covered by two same open subsets, then choose p and q in the intersection respectively, it is easy to see their integrals are equal. □

This leads to the following definition:

Definition 6.1.7 Let ω be a closed 1-form on a topological space X, the *homomorphism of periods*: $\pi_1(X, x_0) \to \mathbb{R}$ is defined as:

$$[\gamma] \mapsto \int_\gamma \omega,$$

where $\gamma : [0,1] \to X$ is a loop represent a homotopy class of $\pi_1(X, x_0)$ with base point $x_0 = \gamma(0) = \gamma(1)$.

Lemma 6.1.8 Suppose X is a locally path-connected topological space, then a continuous closed 1-form $\omega = \mathrm{d}f$ for some continuous real function $f : X \to \mathbb{R}$ on X if and only if the homomorphism of periods determined by ω is trivial for any choice of base point $x_0 \in X$. $\square$

A continuous closed 1-form ω represents a cohomology class $[\omega] \in H^1(X;\mathbb{R}) = \mathrm{Hom}(H_1(X);\mathbb{R})$ by the homomorphism of periods. Lemma 6.1.8 tells us that the homomorphism of periods is independent of the choice of forms up to their cohomology class, two continuous closed 1-forms ω_1, ω_2 lie in the same cohomology class $[\omega_1] = [\omega_2]$ if and only if they differ by an exact form $\mathrm{d}f$ of some real function $f : X \to \mathbb{R}$.

For the purpose of our thesis, we are mainly interested in more specific spaces, e.g. finite CW complexes and compact manifolds, and hence we adapt Lemma 10.5 in [13] to the following form:

Lemma 6.1.9 Let X be a finite CW complex, then any singular cohomology class $\xi \in H^1(X;\mathbb{R})$ can be realized by a continuous closed 1-form on X. $\square$

Remark 6.1.10 Lemma 10.5 in [13] works for more general spaces.

Now we have adequate vocabulary to introduce the concept of category with respect to a closed 1-form: let X be a finite CW complex and ω be a continuous 1-form on X, we introduce the concept of N movability of a subset of X for an integer $N \in \mathbb{Z}$ as follows:

Definition 6.1.11 Let $N \in \mathbb{Z}$ be an integer and $C > 0$ be a real positive constant, a subset $D \subset X$ is N-*movable* with control C with respect to ω if there exists a homotopy $h : D \times [0,1] \to X$ such that h_0 is the inclusion map, and for any $x \in D$ we have

$$\int_x^{h_1(x)} \omega \leq -N,$$

and for all $t \in [0,1]$,

$$\int_x^{h_t(x)} \omega \leq C.$$

We will use the shorthand notation (N, C)-*movable* for a subset with this property. Intuitively, an (N, C)-movable subset is a collection of points which can be continuously deformed in X under a homotopy so that each of them is winding around over distance N. The control C is just a measurement to ensure such deformation does not relapse dramatically during its course. A trivial example can be a null-homotopic subset of X, which firstly contracts to a point and then this point will move around freely under a well-defined homotopy, provided ω is not exact. The following figure shows a case when a subset D is not (N, C)-movable:

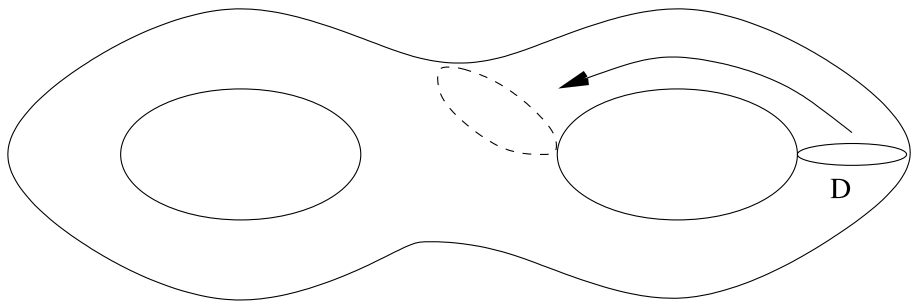

Figure 6.2: Counter-example of (N, C)-movability

Definition 6.1.12 Let X be a finite CW complex and ω be a continuous closed 1-form on X with its cohomology class denoted as $\xi = [\omega]$, then *a Lusternik-Schnirelmann category with respect to closed 1-form* ω, $\operatorname{cat}(X, \xi)$, is defined to be the smallest integer k such that there exists $C > 0$ such that for any integer $N > 0$, there exists an open cover of X, $X = U \cup U_1 \cup \cdots \cup U_k$ such that $U_i \hookrightarrow X$ is null-homotopic in X for $1 \leq i \leq k$ and U is (N, C)-movable with respect to ω.

This is the controlled version introduced in [14], in order to generalise the product inequality of the L-S category. The original version of Farber's category can be obtained by letting $C = \infty$. In fact, it is still an open question whether there is any difference between the controlled category and the ordinary category with respect to a closed 1-form, please see [14]. Nevertheless, the controlled version is useful to study the lower bound of a product space with independent 1-forms for each components, and it is also essential for us to obtain the inequality in the end of this subsection.

For a more detailed exposition of $\mathrm{cat}(X, \xi)$, we refer to [13, Chapter 10], and now we define the relative version of the above category.

Let (X, B) be a CW pair, and ω be a continuous closed 1-form on X.

Definition 6.1.13 Let $N \in \mathbb{Z}$ be an integer and $C > 0$ be a real positive constant, a subset $D \subset X$ containing B is (N, C)-*movable relative to* B with respect to ω if there exists a homotopy $h : D \times [0, 1] \to X$ such that h_0 is the inclusion map, $h_t(B) \subset B$ for all $t \in [0, 1]$ and for any $x \in D$, either $h_1(x) \in B$, or we have

$$\int_x^{h_1(x)} \omega \leq -N,$$

and for all $t \in [0, 1]$,

$$\int_x^{h_t(x)} \omega \leq C.$$

Since we are mainly interested in the relative version in the rest of the chapter, by slight abuse of the notation, we call this property (N, C)-movable without the reference to subset B, unless it causes confusion. Roughly speaking, it is a subset containing B which can be continuously deformed in the space X, such that a point either is pushed into B or travels over distance N as measured by ω.

Definition 6.1.14 Let (X, B) be a finite CW pair and ω be a continuous closed 1-form on X with its cohomology class denoted as $\xi = [\omega]$, then *a relative Lusternik-Schnirelmann category with respect to closed 1-form* ω, $\mathrm{cat}(X, B, \xi)$, is defined to be the smallest integer k such that there exists $C > 0$ such that for any integer $N > 0$, there exists an open cover of X, $X = U \cup U_1 \cup \cdots \cup U_k$ such that $U_i \hookrightarrow X$ is null-homotopic in X for $1 \leq i \leq k$ and U is (N, C)-movable relative to B.

Remark 6.1.15 Note that $\mathrm{cat}(X, B, \xi)$ is independent of ω in the cohomology class $\xi = [\omega]$. For any two closed 1-forms ω, ω' in the same cohomology classes ξ differ by an exact form $\mathrm{d}f$ where $f : X \to \mathbb{R}$, since X is finite, f is bounded above by a real number, say $K \in \mathbb{R}$. then for any path $\gamma : [0, 1] \to X$,

$$|\int_\gamma \omega - \int_\gamma \omega'| = |f(\gamma(1)) - f(\gamma(0))| \leq K,$$

So for any arbitrary $N \in \mathbb{R}$, if there exists an open cover of X with respect to ω, which consists an (N, C)-movable subset with respect to ω and k null-homotopic subsets, then for the same N, we can find an open cover with an (N, C')-movable subset and k null-homotopic subsets for ω'.

Remark 6.1.16 When $B = \emptyset$ is empty, our $\operatorname{cat}(X, B, \xi)$ coincides with $\operatorname{cat}(X, \xi)$: $\operatorname{cat}(X, B, \xi) = \operatorname{cat}(X, \xi)$.

Remark 6.1.17 When the cohomology class is trivial $\xi = 0$, the (N, C)-movable component can't satisfy $\int \omega \leq -N$ for large N, so must have $h_1(X) \in B$ for all $x \in U$, hence our category is equal to the relative version of the classical L-S category, $\operatorname{cat}(X, B, \xi) = \operatorname{cat}(X, B)$. The notion $\operatorname{cat}(X, B)$ can be found in a number of papers, see for instance: [37], [6] and [30].

First of all, we want to state that this category is a homotopy invariant, the proof is similar to the absolute one in Chapter 10.2 of [13].

Lemma 6.1.18 Let $\phi : (X, B) \to (X', B')$ be a relative homotopy equivalence between finite CW-complex pairs (X, B) and (X', B'), and $\xi' \in H^1(X'; \mathbb{R})$, $\xi = \phi^*(\xi') \in H^1(X; \mathbb{R})$, then

$$\operatorname{cat}(X, B, \xi) = \operatorname{cat}(X', B', \xi').$$

Proof: Let $\psi : (X', B') \to (X, B)$ be the relative homotopy inverse of ϕ, ω be a topological closed 1-form on X, and $\omega' = \psi^*(\omega)$ be the closed 1-form on X', then there exists a homotopy $r : (X, B) \times [0, 1] \to (X, B)$ such that, $r_0 = \mathrm{id}$, $r_1 = \psi\phi$. By the compactness of X, there exists $K > 0$ such that

$$\int_x^{r_1(x)} \omega \leq K, \text{ for all } x \in X$$

Now suppose $\operatorname{cat}(X', B', \xi') = k$, i.e., there is some positive real $C > 0$ and for any integer $N > 0$,there is an open cover $X' = V \cup V_1 \cup \cdots \cup V_k$ such that, $V_i \to X'$ are all null-homotopic, and V is $(N - K, C)$-movable relative to B'. In particular, there exists a homotopy $h : V \times [0, 1] \to X'$, so that for any point $x \in V$, either

$h_1(x) \in B'$ or

$$\int_x^{h_1(x)} \omega' \leq -N - K$$

and

$$\int_x^{h_t(x)} \omega \leq C.$$

Let $U = \phi^{-1}(V), U_i = \phi^{-1}(V_i)$, then $X = U \cup U_1 \cup \cdots \cup U_k$, and since the following diagram commutes up to homotopy,

$$\begin{tikzcd} U_i \arrow[r, "\subset"] \arrow[d, "\phi"'] & X \\ V_i \arrow[r, "\subset"] & X' \arrow[u, "\psi"'] \end{tikzcd}$$

we have $U_i \to X$ null homotopic for all i. Hence the proof is reduced to find a homotopy $g : U \times [0,1] \to X$ to move U. Define g_t to be

$$g_t(x) = \begin{cases} r_{2t}(x) & 0 \leq t \leq \frac{1}{2} \\ \psi \circ h_{2t-1} \circ \phi(x) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then g is continuous since both r_t and $\psi \circ h_{2t-1} \circ \phi$ are continuous for all t and agree at $t = \frac{1}{2}$. Moreover, $g_0 = r_0|_U$ is the inclusion, and $g_t(B) \subset B$ since both $r_t(B) \subset B$ and $\psi \circ h_{2t-1} \circ \phi(B) \subset B$ for all t. Finally, for any $x \in U$, either $g_1(x) \in B$ or

$$\int_x^{g_1(x)} \omega = \int_x^{r_1(x)} \omega + \int_{r_1}^{\psi(h_1(\phi(x))} \omega \leq K + (-N - K) = -N$$

and

$$\int_x^{g_t(x)} \omega < K + C, \text{ for all } t \in [0,1].$$

So it is true that $\mathrm{cat}(X, B, \xi) \leq \mathrm{cat}(X', B', \xi')$. Repeat the same argument to get $\mathrm{cat}(X', B', \xi') \leq \mathrm{cat}(X, B, \xi)$, we have proved the lemma. □

6.2 Properties

We detail the features of the relative category with respect to a closed 1-form in the following two subsections, namely, an inequality that links the relatives categories and a lower bound estimate.

6.2.1 Inequalities

Let us firstly state the main inequality for relative categories:

Theorem 6.2.1 Let $A \subset B \subset X$ be finite CW complexes and $\xi \in H^1(X;\mathbb{R})$ be a cohomology class of X, then

$$\mathrm{cat}(X, A, \xi) \leq \mathrm{cat}(X, B, \xi) + \mathrm{cat}\big(B, A, i^*(\xi)\big)$$

where the map $i^* : H^1(X;\mathbb{R}) \to H^1(B;\mathbb{R})$ is induced by the inclusion map $i : B \to X$.

Proof: Suppose $\mathrm{cat}(X, B, \xi) = k$ and $\mathrm{cat}(B, A, i^*(\xi)) = l$, let ω be a continuous closed 1-form representing ξ, we need to show the existence of a real positive $R > 0$, such that for any $N > 0$, there is an open cover of X which consists of $k + l$ null-homotopic components and one (N, R)-movable component relative to A.

Firstly, we want to modify the open cover of B to be open in X, we do the following trick of deformation retraction:

According to Hatcher [17, Appendix A.2], there exists an open neighbourhood $N(B)$ of B in X such that there exists a deformation retraction $D' : \overline{N(B)} \times [0,1] \to \overline{N(B)}$ rel B with $D'_1(\overline{N(B)}) = B$. And we can extend its composition with the inclusion map $\overline{N(B)} \times [0,1] \to X$ to the whole space, denote $D : X \times [0,1] \to X$ with $D_t|_{\overline{N(B)}} = D'_t$ for all t, according to Example 0.15 of [17]. By the compactness of X, there exists $K \in \mathbb{R}$ such that $\int_x^{D_1(x)} \omega < K$ for any $x \in X$.

Now according to the definition of the category, there is $C > 0$ and for any integer N, there exist open covers $X = U \cup U_1 \cup \cdots \cup U_k$ and $B = V \cup V_1 \cup \cdots \cup V_l$, where U_i and V_j are null-homotopic for all i, j; U is $(N + C + 1 + K, C)$-movable relative to B by a homotopy g, and V is $(N + C + 2K)$-movable relative to A by a homotopy h.

On the other hand, as N varies, $\overline{N(B)}$ is not necessarily contained in U for all $N > 0$, therefore, let us consider the intersection $N'(B) = N(B) \cap U$ of $N(B)$ and U and restrict the deformation retraction to the closure of this intersection as $d = D|_{\overline{N'(B)}} : \overline{N'(B)} \times [0,1] \to X$. Note we still have $K > 0$ with $\int_x^{d_1(x)} \omega < K$ for any $x \in \overline{N'(B)}$. Also denote $N''(B)$ an open subset of $N'(B)$ with $N''(B) \subset \overline{N''(B)} \subset N'(B)$, in particular, $N''(B) \subset \big(d_1^{-1}(V) \cup d_1^{-1}(V_1) \cup \cdots \cup d_1^{-1}(V_l)\big)$.

Secondly, to comply with the definition of relative movability, let us modify $g : U \times [0,1] \to X$ such that points in A stay in A throughout the homotopy. Now according to the Lemma 6.2.2 below, we can construct an open neighbourhood $N(A)$ of A in U with $g_t(N(A)) \subset N(B) \cap U$ for all $t \in [0,1]$. Then we define map $\varphi : U \to [0,1]$ such that $\varphi|_A = 0$ and $\varphi|_{U-N(A)} = 1$. So we have a continuous homotopy $g' : U \times [0,1] \to X$ as

$$g'(x,t) = D(g(x, \varphi(x)t), t),$$

so $g'_t(N(A)) \subset N(B) \cap U$ as g_t does for all t, and for any $x \in U$, either $g'_1(x) \in B$ or $\int_x^{g'_1(x)} \omega < -N - C - 1$ and for all $x \in U$ and all $t \in [0,1]$, $\int_x^{g'_t(x)} \omega < C + K$ for some $C \in \mathbb{R}$.

Now we want to show there is an open cover of X modified from the ones of X and B, namely:

$$X = (U^* \cup V^*) \cup (U_1^* \cup \cdots \cup U_k^*) \cup (V_1^* \cup \cdots \cup V_l^*),$$

where $U^* \cup V^*$ is (N, R)-movable relative to A for some $R > 0$ and U_i, V_j are null-homotopic in X.

We divide the argument into three parts:

(i) **Null homotopy of V_j^*** To get V_j^*, we firstly need to modify the V_j's so that they are open in X. Since d is continuous, we have $\tilde{V}_j = d_1^{-1}(V_j) \subset N'(B)$ is open in X. Now we set $V_j^* = (g'_1)^{-1}(\tilde{V}_j)$ and define the null homotopy $H_j : V_j^* \times [0,1] \to X$ as

$$H_j(x,t) = \begin{cases} g'(x, 3t) & 0 \le t \le \frac{1}{3} \\ d(g'_1(x), 3t-1) & \frac{1}{3} \le t \le \frac{2}{3} \\ h_j(d_1 g'_1(x), 3t-2) & \frac{1}{2} \le t \le 1, \end{cases}$$

where h^j is the null homotopy of V_j, and we can see H_j^t continuously deform V_j^* to a point in X.

Notice that V_j^* does not necessarily contain the original V_j but it encompasses the points of U that reach $\tilde{V}_j$ after flowing along g_t.

(ii) **Construction of** V^* Here we want to modify V and the accompanied homotopy h so that the new V^* is open in X and $(N+C+K, C+1)$-movable relative to A by some homotopy. Consider the closed complement $V^c = B - \cup_j V_j$ in B, we have $d_1^{-1}(V^c)$ closed in X and thus denote $\tilde{V}^c = d_1^{-1}(V^c) \cap \overline{N''(B)}$ a closed subset in X. Meanwhile, denote $\tilde{V} = d_1^{-1}(V) \subset N'(B)$ which is open in X with $\tilde{V}^c \subset \tilde{V}$. Notice also that there exists homotopy $h' : \tilde{V} \times [0,1] \to X$ for $\tilde{V}$ as:

$$h'(x,t) = \begin{cases} d(x, 2t) & 0 \le t \le \frac{1}{2} \\ h(d_1(x), 2t-1) & \frac{1}{2} \le t \le 1, \end{cases}$$

such that for $x \in \tilde{V}$,

$$\text{either } h_1'(x) \in A \text{ or } \int_x^{h_1'(x)} \omega < -N - C - 2K + K = -N - C - K;$$

and $\int_x^{h_t'(x)} \omega < C + K$ for all $x \in \tilde{V}$ and $t \in [0,1]$.

Now according to Lemma 6.2.3 below, there exists an open subset V' of X with $\tilde{V}^c \subset V' \subset \tilde{V}$ and a homotopy $H : X \times [0,1] \to X$ such that:

$$H_0(x) = x \text{ for all } x \in X;$$

and for all $x \in V'$, either $H_1(x) \in A$ or

$$\int_x^{H_1(x)} \omega \le -N - C - K$$

and for all $x \in X$ and all $t \in [0,1]$

$$\int_x^{H_t(x)} \omega < C + 1.$$

We set $V^* = (g_1')^{-1}(V')$.

(iii) **Construction of** U^* Choose slightly smaller open subsets $U_i^o \subset U_i$ such that:

$$U_i^o \subset \overline{U_i^o} \subset U_i \text{ and } X \subset U \cup U_1^o \cup \cdots \cup U_k^o,$$

then we define

$$U^* = X - \left((\bigcup_{i=1}^{k} \overline{U^o}_i) \cup (g_1')^{-1}(\overline{N''(B)}) \right).$$

Define the homotopy $G : \big(U^* \cup V^*\big) \times [0,1] \to X$ as:

$$G(x,t) = \begin{cases} g'(x,2t) & 0 \le t \le \frac{1}{2} \\ H(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

Check that $G_t(A) \subset A$ for all $t \in [0,1]$ as both g' and H are built with this feature. For $x \in U^*$ it will travel over N distance as:

$$\int_x^{G_1(x)} \omega = \int_x^{g_1'(x)} \omega + \int_{g'(x)}^{H_1(x)} \omega \le (-N - C - 1) + (C+1) = -N.$$

Similarly, for $x \in V^* = (g_1')^{-1}(V')$, after discounting the effect of g' and returning into $V' \in N(B)$, H either pushes the point into A or to travel over N distance as

$$\int_x^{G_1(x)} \omega = \int_x^{g_1'(x)} \omega + \int_{g'(x)}^{H_1(x)} \omega \le C + K + (-N - C - K) = -N.$$

also for all $t \in [0,1]$ and $x \in U^* \cup V^*$, $\int_x^{G_t(x)} \omega < 2C + 2K + 1$.

Finally, let us set $U_i^* = U_i$ unchanged, then X is covered as:

$$X = (U^* \cup V^*) \cup (U_1^* \cup \cdots \cup U_k^*) \cup (V_1^* \cup \cdots \cup V_l^*).$$

This is true as $(g_1')^{-1}(\overline{N''(B)})$ is covered by V^* and V_j^*:

$$(g_1')^{-1}(\overline{N''(B)}) \subset V^* \cup V_1^* \cup \cdots \cup V_l^*,$$

where

$$\begin{aligned} \overline{N''(B)} \quad & \big(\subset \; d_{\frac{1}{2}}^{-1}(V^c) \cup d_1^{-1}(V_1) \cup \cdots \cup d_1^{-1}(V_l) \big) \\ & \subset \; V' \cup \tilde{V}_1 \cup \cdots \cup \tilde{V}_l; \end{aligned}$$

and $\{U_i^*\}$ covers the rest of X.

Now $U^* \cup V^*$ is $(N, 2C+2K+1)$-movable relative to A and the other components are all null-homotopic. $\square$

Lemma 6.2.2 Using the notation as above, there exists an open neighbourhood $N(A)$ of A in X with $N(A) \subset N(B) \cap U$, such that $g_t(N(A)) \subset N(B) \cap U$ for all $0 \leq t \leq 1$.

Proof: Given $g : U \times [0,1] \to X$, we have $g_t(a) \in B \subset N(B) \cap U$ for any $a \in A$, according to the hypothesis. For such point $(a,t) \in A \times [0,1]$, by the continuity of g, we can find some neighbourhood $N^t(a) \times (t - \delta_t, t + \delta_t)$ of (a,t) in X for small δ_t, such that $g(a', t') \in N(B) \cap U$ for all $(a', t') \in N^t(a) \times (t - \delta_t, t + \delta_t)$. Note that the size of the neighbourhood $N^t(a)$ here depends on t.

Now because of the compactness of $[0,1]$, the open cover $\bigcup_{t \in [0,1]} (t - \delta_t, t + \delta_t)$ of $[0,1]$ induces a finite subcover, namely, there exists $t_1, \ldots, t_n$ such that $[0,1] = \bigcup_{i=1}^{n} (t_i - \delta_i, t_i + \delta_i)$. Set

$$N(a) = \bigcap_{i=1}^{n} N^{t_i}(a),$$

we claim $g(N(a) \times [0,1]) \in N(B) \cap U$.

For any $a \in A$, this is true as let any $(x,t) \in N(a) \times [0,1]$ then $t \in (t_i - \delta_i, t_i + \delta_i)$ for some i, and $x \in N(a) \subset N^{t_i}(a)$, i.e. $(x,t) \in N^{t_i}(a) \times (t_i - \delta_i, t_i + \delta_i)$ for some i, which satisfies $g(x,t) \in N(B) \cap U$ by construction.

Now define

$$N(A) = \bigcup_{a \in A} N(a).$$

We can see $N(A) \subset N(B) \cap U$ and $g_t(N(A)) \subset N(B) \cap U$ for all $t \in [0,1]$. $\square$

Now we justify the lemma used in the proof of Theorem 6.2.1 for the extension of the homotopy h to the whole complex X:

Lemma 6.2.3 Let ω be a continuous closed one form on a finite CW complex X. Let $B \subset X$ be a subcomplex. Suppose further that there exists $C \in \mathbb{R}$ so that for an integer $N > 0$, we have an open subset U of X containing B and U is N-movable with respect to B. Then for any given closed subset $W \subset U$ with $B \subset W$, there exists an open set U' with $W \subset U' \subset U$ and a homotopy $H : X \times [0,1] \to X$ satisfying the following properties:

1. $H_0(x) = x$ for all $x \in X$ and $H_t(B) \subset B$ for all $t \in [0,1]$;

2. For any $x \in U'$ one has either $H_1(x) \in B$ or $\int_x^{H_1(x)} \omega < -N$;

3. For any $x \in X$ and $t \in [0,1]$, $\int_x^{H_t(x)} \omega < C + 1$.

The lemma states the validity of extending the homotopy for a relatively N-movable set to the whole space X, so that a smaller open set remains N-movable relative to B, whereas the control condition holds for the whole X under such homotopy.

Proof: Let $h : U \times [0,1] \to X$ be the original homotopy for U in the hypothesis.

Then we want to enforce the control condition on the whole space X. To do this, we denote a smaller open neighbourhood U' of W with $W \subset U' \subset U''$, then use a bump function $\phi : X \to [0,1]$ such that

$$\phi(x) = \begin{cases} 1 & x \in W \\ 0 & x \in X - U'. \end{cases}$$

Then we can define $H : X \times [0,1] \to X$ as

$$H(x,t) = h(x, \phi(x)t.$$

The homotopy H satisfies the requirements. □

If $A = \emptyset$ is empty, Theorem 6.2.1 states the connection of the absolute category and the relative category in an inequality as in the following corollary:

Corollary 6.2.4 For a CW pair (X,B) and a continuous closed 1-form ω on X, we have

$$\mathrm{cat}(X,\xi) \leq \mathrm{cat}(X,B,\xi) + \mathrm{cat}\big(B, i^*(\xi)\big).$$

where $\xi = [\omega]$ is the cohomology class of ω, and $i^* : H^1(X;\mathbb{R}) \to H^1(B;\mathbb{R})$ is induced by the inclusion map $i : B \to X$. □

Assisted by Lemma 6.2.3 above, we can also derive a similar inequality for the category of a product of CW complex pairs, see [14].

Theorem 6.2.5 Let $(X,B),(Y,D)$ be two CW pairs, $\xi_X \in H^1(X;\mathbb{R})$ and $\xi_Y \in H^1(Y;\mathbb{R})$ be the cohomology classes on X and Y, respectively. Suppose also

$$\mathrm{cat}(X,B,\xi_X) > 0 \qquad \text{or} \qquad \mathrm{cat}(Y,D,\xi_Y) > 0,$$

Then

$$\mathrm{cat}((X,B)\times(Y,D),\xi)\leq \mathrm{cat}(X,B,\xi_X)+\mathrm{cat}(Y,D,\xi_Y)-1,$$

with $\xi=\xi_X\times 1+1\times\xi_Y$.

Proof: Suppose $\mathrm{cat}(X,B,\xi_X)=k$ and $\mathrm{cat}(Y,D,\xi_Y)=l$. There is $C>0$ and for any N let the open covers for X and Y be

$$X=U\cup U_1\cup\cdots\cup U_k$$

and

$$Y=V\cup V_1\cup\cdots\cup V_l$$

respectively. Here U and V are $(N+C+1,C)$-movable relative to B and D respectively.

Now for the closed subsets $W_X=X-\bigcup_{1\leq i\leq k}U_i$ and $W_Y=Y-\bigcup_{1\leq j\leq l}V_j$ of X and Y respectively, Lemma 6.2.3 above provides us with the existence of global homotopies H_X and H_Y, so that open neighbourhoods U' and V' of W_X and W_Y with $W_X\subset U'\subset U$ and $W_Y\subset V'\subset V$ are $(N+C+1,C+1)$-movable.

Hence, the open product subset $U'\times V'\subset U\times V$ contains $W_X\times Y\cup X\times W_Y$ and is $(N,C+1)$-movable.

There for $U'\times V'$ together with $U_i\times V_j$ cover $X\times Y$. And results in [7] and [19] state the fact that the union of $U_i\times V_j$ can be covered by $l+k-1$ null-homotopic open subsets in $X\times Y$. Therefore the inequality holds. □

6.2.2 Cup length lower bound

We now provide a cohomology lower bound for $\mathrm{cat}(X,B,\xi)$ similar to the one in [14], let us begin with some basic notions.

For a CW complex X and a continuous closed 1-form ω, we have a regular covering space $p:\tilde{X}\to X$ correspond to the kernel of the cohomology class $\xi=[\omega]\in H^1(X;\mathbb{R})$, then we have covering transformation group $H\simeq\mathbb{Z}^r=\pi_1(X)/\ker(\xi)$. Then the cohomology class of the pullback of ω is trivial in the covering, $[p^*\omega]=0\in H^1(\tilde{X};\mathbb{C})$, i.e., there exists a real function $f:\tilde{X}\to\mathbb{R}$ such that $\mathrm{d}f=p^*\omega$.

Definition 6.2.6 A subset $O \subset X$ is called a *neighbourhood of infinity* in $\tilde{X}$ with respect to a cohomology class $\xi \in H^1(X;\mathbb{R})$, if O contains the set $\{x \in \tilde{X} : f(x) < c\}$ for some $c \in \mathbb{R}$. Here $f : \tilde{X} \to \mathbb{R}$ is the real function obtained by pulling back a closed 1-form ω with $[\omega] = \xi \in H^1(X;\mathbb{C})$.

Notice the definition of a neighbourhood of infinity O is independent of the choice of real functions, the argument goes similarly as for the independency of movable open subset to the choice of closed 1-forms up to the cohomology class. Namely, for two functions $f, f' : \tilde{X} \to \mathbb{R}$, they can be pushed forwardly to two closed 1-forms in the base space as ω and ω' respectively. And there is a real function $g : X \to \mathbb{R}$ such that $\omega = \omega' + \mathrm{d}g$. Then $f = f' + g \cdot p$, and because X is compact, there is $K \in \mathbb{R}$ with $g(X) < K$. Therefore, the neighbourhood of infinity O defined by f for some $c \in \mathbb{R}$ can also be defined by f' by $c + K$.

For a more detailed exposition of the above notion, we refer to [15], in particular Lemma 3 in Section 3.

Definition 6.2.7 Let (X, B) be a finite CW complex pair and ω be a continuous closed 1-form on X. Suppose $p : \tilde{X} \to X$ is a regular covering corresponding to $\ker(\xi)$ where $\xi = [\omega] \in H^1(X)$ is the cohomology class of ω. Then a homology class $z \in H_i(\tilde{X}, \tilde{B})$ is *movable to infinity with respect to* ξ, if in any neighborhood O of infinity with respect to ξ, z is the image of a homology class in $H_i(O, O \cap \tilde{B})$ under the map $H_i(O, O \cap \tilde{B}) \to H_i(\tilde{X}, \tilde{B})$.

Notation 6.2.8 Let $H = H_1(X;\mathbb{Z})/\ker(\xi)$, denote $\mathcal{V}_\xi = (\mathbb{C}^*)^r = \mathrm{Hom}(H, \mathbb{C}^*)$. We can think of $\mathcal{V}_\xi$ as the variety of all complex flat line bundles L over X such that the induced flat line bundle p^*L on $\tilde{X}$ is trivial.

Definition 6.2.9 In $\mathcal{V}_\xi$ a bundle L is called ξ*-transcendental* if the monodromy $\mathrm{Mon}_L : \mathbb{Z}[H] \to \mathbb{C}$ is injective, and ξ*-algebraic* if not.

Notice that if L is ξ-transcendental, the covering space $p : \tilde{X} \to X$ induces maps $p_* : H_*(\tilde{X}, \tilde{B}; \mathbb{C}) \to H_*(X, B; L)$ and $p^* : H^*(X, B; L) \to H^*(\tilde{X}, \tilde{B}; \mathbb{C})$, because $p^*(L) = \mathbb{C}$.

The following two assertions are the relative versions of Proposition 6.4 and Theorem 4 in [14], their validity follows similar algebraic arguments provided in [14].

Proposition 6.2.10 Suppose $L \in \mathcal{V}_\xi$ is ξ-transcendental, and $v \in H^q(X, B; L)$ is a non-zero cohomology class. Then there exists a homology class $z \in H_q(\tilde{X}, \tilde{B}; \mathbb{C})$ with $v \frown p_*(z) \neq 0$. □

Theorem 6.2.11 Suppose a flat line bundle is ξ-transcendental and there is cohomology class $v \in H^q(X, B; L)$ with $v \frown p_*(z) \neq 0$ for some $z \in H_q(\tilde{X}, \tilde{B}; \mathbb{C})$ and $p_*(z) \in H_q(X, B; L^*)$, where L^* is the dual bundle of L. Then z is not movable to infinity with respect to ξ. □

We now state the cohomology estimate of the category:

Theorem 6.2.12 Suppose $L \in \mathcal{V}_\xi$ is ξ-transcendental, and $v_0 \in H^{d_0}(X, B; L), v_1 \in H^{d_0}(X; \mathbb{C}), \ldots, v_k \in H^{d_k}(X; \mathbb{C})$ with $d_i > 0$ for all $i = 0, \ldots, k$ are such that

$$v_0 \smile v_1 \smile \cdots \smile v_k \neq 0 \in H^d(X, B; L), \tag{6.1}$$

with $d = \Sigma_i d_i$, then

$$\mathrm{cat}(X, B, \xi) > k$$

The maximal k gives a lower bound for $\mathrm{cat}(X, b, \xi)$, and it gives a *cup length* estimate for $\mathrm{cat}(X, B, \xi)$.

Proof: Let $v = v_1 \smile \cdots \smile v_k$, according to (6.1) and Proposition 6.2.10, we can find a homology class $z \in H_d(\tilde{X}, \tilde{B}; \mathbb{C})$ such that

$$(v_0 \smile v) \frown p_*(z) \neq 0.$$

Fix such a homology class $z \in H_d(\tilde{X}, \tilde{B}; \mathbb{C})$, then it is possible to choose a compact polyhedron $K \subset \tilde{X}$ such that z is the image of some homology class in $H_d(K, \tilde{B} \cap K; \mathbb{C})$ under the inclusion-induced map $i_* : H_d(K, \tilde{B} \cap K; \mathbb{C}) \to H_d(\tilde{X}, \tilde{B}; \mathbb{C})$. We denote this homology class z' in $H_d(K, \tilde{B} \cap K; \mathbb{C})$. Now we assert the existence of a neighbourhood of infinity O_∞ which possesses the following property: if the image of a homology class under the map $H_*(K, \tilde{B} \cap K; \mathbb{C}) \to H_*(\tilde{X}, \tilde{B}; \mathbb{C})$ has a preimage in $H_*(O_\infty, O_\infty \cap \tilde{B}; \mathbb{C})$, then it is movable to infinity. Indeed, let

$O = f^{-1}((-\infty, 0]) \subset \tilde{X}$ be a neighbourhood of infinity, and $g : \tilde{X} \to \tilde{X}$ be a covering transformation such that $\xi(g) < 0$. Then

$$V_g = \mathrm{Im}\,[H_*(gO, gO \cap \tilde{B}; \mathbb{C}) \to H_*(\tilde{X}, \tilde{B}; \mathbb{C})] \cap \mathrm{Im}\,[H_*(K, \tilde{B} \cap K; \mathbb{C}) \to H_*(\tilde{X}, \tilde{B}; \mathbb{C})]$$

is a finite dimensional complex vector space.

Now by the choice of g, we have $g^n O \subset gO$ for any $n > 0$. Since the inclusion-induced map $H(g^n O, g^n O \cap \tilde{B}; \mathbb{C}) \to H_*(\tilde{X}, \tilde{B}; \mathbb{C})$ factors through $H(g^n O, g^n O \cap \tilde{B}; \mathbb{C}) \to H(gO, gO \cap \tilde{B}; \mathbb{C})$, hence $V_{g^n} \subseteq V_g$. Now finite dimensionality of the vector space V_g implies the following chain

$$\cdots \subset V_{g^n} \subset \cdots \subset V_{g^2} \subset V_g \subset V$$

stabilises after finitely many terms. Subsequently, there exists a sufficiently large $N > 0$ such that $V_{g^n} = V_{g^N}$ for any $n \geq N$. Therefore, fix such a N and the subset $O_\infty = g^N O$ will work.

So let us have such a neighbourhood O_∞, then the pullback function $f : \tilde{X} \to \mathbb{R}$ of ω with $p^*\omega = \mathrm{d}f$ gives values to points in K and O_∞. In particular, we have $f(K) \subset [a, b]$ and $O_\infty \supset f^{-1}(-\infty, c)$, for some $c < a < b$. Note that $c < a$ is always available by increasing N if necessary.

Now assume the statement is false, then $\mathrm{cat}(X, B, \xi) \leq k$, in particular, for $N > b - c$ and some $C > 0$, there exists an open cover of X:

$$X = U \cup U_1 \cup \cdots \cup U_k,$$

where $U_i \hookrightarrow X$ is null-homotopic and U is (N, C)-movable relative to B.

Now observe that $v_i \in H^{d_i}(X; \mathbb{C})$ can be pulled back to some $u_i \in H^{d_i}(X, U_i; \mathbb{C})$ because of the null-homotopy of U_i.

Therefore, by naturality of the cup product, $v = j^*(u) \in H^{d-d_0}(X; \mathbb{C})$ for some $u = u_1 \smile \cdots \smile u_k \in H^{d-d_0}(X, U_1 \cup \cdots \cup U_k; \mathbb{C})$, where j^* is induced by inclusion $j : (X, \emptyset) \to (X, U_1 \cup \cdots \cup U_k)$.

Let w be the image of $p^*(u)$ via the inclusion-induced map

$$i_1^* : H^{d-d_0}(\tilde{X}, \tilde{U}_1 \cup \cdots \cup \tilde{U}_k; \mathbb{C}) \to H^{d-d_0}(K, (\tilde{U}_1 \cup \cdots \cup \tilde{U}_k) \cap K; \mathbb{C}),$$

and restrict the lift $(\tilde{X}, \emptyset) \to (\tilde{X}, \tilde{U}_1 \cup \cdots \cup \tilde{U}_k)$ of j to K as:

$$\tilde{\jmath} : (K, \emptyset) \to (K, (\tilde{U}_1 \cup \cdots \cup \tilde{U}_k) \cap K),$$

then

$$\tilde{\jmath}^* w \frown z' \in H_{d_0}(K, \tilde{B} \cap K),$$

where $\tilde{\jmath}^* w \in H^{d-d_0}(K; \mathbb{C})$ and $z' \in H_d(K, \tilde{B} \cap K; \mathbb{C})$. Notice $\tilde{\jmath}^* w \frown z' \neq 0$ as by naturality of the cap product, see [41, Lemma 5.6.16, pp 254] for instance,

$$\begin{aligned} i_*(\tilde{\jmath}^* w \frown z') &= i_*(\tilde{\jmath}^* i_1^*(p^*(u)) \frown z') = i_*((i_1 \tilde{\jmath})^*(p^*(u)) \frown z') \\ &= p^*(u) \frown i_{2*}(z') = p^*(u) \frown j_{1*} i_*(z') \\ &= j_2^* p^*(u) \frown i_*(z') = (p j_2)^*(u) \frown z \\ &= (jp)^*(u) \frown z = p^*(j^*(u)) \frown z \\ &= p^*(v) \frown z, \end{aligned}$$

which is non-trivial according to our hypothesis. Here $i_{2*} : H_d(K, \tilde{B} \cap K; \mathbb{C}) \to H_d(\tilde{X}, \tilde{B} \cup (\tilde{U}_1 \cup \cdots \cup \tilde{U}_k))$ is induced by the inclusion map $j_1 i : (K, \tilde{B} \cap K) \to (\tilde{X}, \tilde{B} \cup (\tilde{U}_1 \cup \cdots \cup \tilde{U}_k))$ with $j_1 : (\tilde{X}, \tilde{B}) \to (\tilde{X}, \tilde{B} \cup (\tilde{U}_1 \cup \cdots \cup \tilde{U}_k))$.

And j_2^* is induced by the inclusion $j_2 : (\tilde{X}, \emptyset) \to (\tilde{X}, \tilde{U}_1 \cup \cdots \cup \tilde{U}_k)$ which commutes with p in the following diagram:

$$\begin{tikzcd} (\tilde{X}, \emptyset) \arrow[r, "j_2"] \arrow[d, "p"] & (\tilde{X}, \tilde{U}_1 \cup \cdots \cup \tilde{U}_k) \arrow[d, "p"] \\ (X, \emptyset) \arrow[r, "j"] & (X, U_1 \cup \cdots \cup U_k) \end{tikzcd}.$$

Again by naturality of the cap product,

$$i'_*(\tilde{\jmath}^* w \frown z') = w \frown \bar{\imath}_*(z),$$

where i'_* is induced by $i' : (K, \tilde{B} \cap K) \to (K, \tilde{U} \cap K)$ and $\bar{\imath}_*$ is from

$$\bar{\imath} : (K, \tilde{B} \cap K) \to (K, \Big((\tilde{U}_1 \cup \cdots \cup \tilde{U}_k) \cap K\Big) \cup \Big(\tilde{U} \cap K\Big)) = (K, K).$$

Therefore, $\bar{\imath}_*(z) = 0 \in H_{d_0}(K, K) = 0$, and $i'_*(\tilde{\jmath}^* w \frown z) \in H_{d_0}(K, \tilde{U} \cap K)$ is trivial. Consequently, the exact sequence

$$\cdots \to H_{d_0}(\tilde{U} \cap K, \tilde{B} \cap K) \to H_{d_0}(K, \tilde{B} \cap K) \xrightarrow{\bar{\imath}_*} H_{d_0}(K, \tilde{U} \cap K) \to \cdots$$

indicates the existence of a nontrivial preimage z_0 of $\bar{\jmath}^* w \frown z$ in $H_{d_0}(\tilde{U} \cap K, \tilde{B} \cap K)$.

Now for the (N, C)-movable open subset U in X, its lift $\tilde{U}$ in $\tilde{X}$ has homotopy $h : (\tilde{U}, \tilde{B}) \times [0,1] \to (\tilde{X}, \tilde{B})$, which yields equality $(h_0)_*(z_0) = (h_1)_*(z_0) \in H_{d_0}(\tilde{X}, \tilde{B})$, where $(h_0)_*(z_0) = i''_*(z_0) \in H_{d_0}(\tilde{X}, \tilde{B})$ with $i'' : (\tilde{U} \cap K, \tilde{B} \cap K) \to (\tilde{X}, \tilde{B})$. And since $h_1 : (\tilde{U} \cap K, \tilde{B} \cap K)$ factors through $(\tilde{U} \cap K, \tilde{B} \cap K) \to (O_\infty, O_\infty \cap \tilde{B})$ due to our choice of $N > b - c$ for $\tilde{U}$, there exists a homology class in $H_{d_0}(O_\infty, O_\infty \cap \tilde{B})$ that maps to $(h_1)_*(z_0)$. In other words, $i''_*(z_0)$ is movable to infinity, a contradiction to the nontrivial value of $i'_*(z_0)$ according to Theorem 6.2.11:

$$\begin{aligned} v_0 \frown p_*(i''_*(z_0)) &= v_0 \frown p_*(i_*(\bar{\jmath}^* w \frown z')) = v_0 \frown p_*(p^*(v) \frown z) \\ &= (v_0 \smile v) \frown p_*(z) \neq 0. \end{aligned}$$

We have shown the proof. □

6.3 Homoclinic cycles and critical points

In this section, we study the critical points of a closed 1-form ω and homoclinic cycles on a manifold M with boundary by the invariant $\mathrm{cat}(M, B, [\omega])$ developed in the preceding sections. Here B is specified according to ω, namely, the exit set of ω. In the manifold setting, we find it more convenient to study the modified category rather than the one defined in the beginning of the chapter, namely, instead of allowing the subset B to move within itself, we fix B:

Definition 6.3.1 Let ω be a continuous closed 1-form on a finite CW complex X, denote its cohomology class as $\xi = [\omega]$, and $B \subset X$ be a closed subset of X, then $\mathrm{cat}^0(X, B, \xi)$ is defined to be the smallest integer k such that there exists $C > 0$ such that for any $N > 0$, there exists an open cover of X, $X = U \cup U_1 \cup \cdots \cup U_k$, satisfying:

a) The inclusion $U_i \to X$ is null-homotopic in X for $1 \leq i \leq k$;

b) The closed subset $B \subset U$ and there exists homotopy $h : U \times [0,1] \to X$ rel B such that $h_0 : U \to X$ is the inclusion map, and for any $x \in U$, either $h_1(x) \in B$, or we have

$$\int_x^{h_1(x)} \omega \leq -N,$$

and for all $t \in [0,1]$,

$$\int_x^{h_t(x)} \omega \leq C.$$

Remark 6.3.2 It turns out that $\mathrm{cat}^0(X,B,\xi) = \mathrm{cat}(X,B,\xi)$, we provide a proof at the end of this section.

Now we specify B here:

Let $\rho : \bar{M} \to M$ be a regular covering space of M with $\pi_1(\bar{M}) = \ker([\omega])$, then there exists a real function $f : \bar{M} \to \mathbb{R}$ with $\mathrm{d}f = \rho^*(\omega)$. We repeat the construction in the beginning of Section 2.1 in Chapter 2. Namely, we equip the boundary ∂M of the M with a tubular neighbourhood structure $\partial M \times [0,1) \subset M$ such that $\partial M \times \{0\} \cong \partial M$, and a point in the tubular neighbourhood will be denoted as $(x,t) \in \partial\bar{M} \times [0,1)$. We denote the lift of this tubular neighbourhood in the covering as $\partial\bar{M} \times [0,1)$. Then for each $x \in \partial M$ the equivariance of $\dfrac{\partial f}{\partial t}$ enables us to have a well-defined partial derivative $\dfrac{\partial f}{\partial t}$ on the base manifold M.

Definition 6.3.3 Given a fixed inner collaring $\partial M \times [0,1)$ of the boundary ∂M in M, *the exit set of* ω is defined as:

$$B = \{x \in \partial M : \frac{\partial f}{\partial t}(x,0) \leq 0\}.$$

Additionally, we need the boundary assumptions for ω on B set up in the beginning of the thesis, again the partial derivative $\dfrac{\partial f}{\partial t}$ is capable of describing such conditions. Recall:

Assumptions on ω on ∂M

A1 The function $\bar{f}$ has no critical point on $\partial\bar{M}$. This implies that $\bar{f}$ has no critical points in a neighbourhood of $\partial\bar{M}$. Without loss of generality we assume that $\bar{f}$ has no critical points in the entire collaring $\partial\bar{M} \times [0,1)$.

A2 The partial derivative $\frac{\partial f}{\partial t}$, where t is the coordinate for $[0,1)$, is a smooth function on $\partial\bar{M} \times \{0\}$, and zero is a regular value of $\frac{\partial f}{\partial t}(x,0)$. Denote by $\Gamma = \{x \in \partial M : \frac{\partial f}{\partial t}(x,0) = 0\}$, this is equivalent to say Γ is a 1-codimensional closed submanifold of ∂M.

A3 Fix a tubular collaring of Γ in $\partial\bar{M}$, $\Gamma \times [-1,1] \subset \partial\bar{M}$, with $\Gamma \times [-1,0] \subset B$. So if a point lies in the cubical neighbourhood of Γ in M, we write it in local coordinates:

$$(x,s,t) \in \Gamma \times [-1,1] \times [0,1),$$

where $x = (x_1, \cdots, x_{m-2})$, then we assume

$$\frac{\partial f}{\partial s}(x,0,0) > 0.$$

By always choosing a product Riemannian metric near the boundary $\partial M \times [0,1)$ and $\Gamma \times [-1,1]$, these assumptions ensure the flow, which is generated by corresponding gradient vector field of ω transverse to ∂M, exits through interior of B transversely, and Corollary 1.2.6 in Chapter 1 ensures the continuity of the time keeping for the emitting flow. Here we restate it in the category context:

Lemma 6.3.4 Let v be a gradient of ω transverse to $(\partial M, B)$ and denote by

$$U_B = \{x \in M : \text{ there exists } t \in \mathbb{R}, \text{ such that } x \cdot t \in B\},$$

then the function $\beta : U_B \to \mathbb{R}$ defined as $\beta(x) = \min\{t : x \cdot t \in B\}$ is continuous, and U_B is open in M. $\square$

Notation 6.3.5 Let $\Phi : \Delta \to M$ be the negative gradient flow of a negative gradient vector field v of ω, where

$$\Delta = \bigcup_{x \in M} \{x\} \times J_x \subset M \times \mathbb{R}$$

with J_x as maximal interval for which differential equation $\gamma'(t) = v(\gamma_x(t))$ is defined for map $\gamma : J_x \to M$ with $\gamma_x(0) = x$. we denote $x \cdot t = \Phi(x,t) \in M$ as the flow line passing x as t varies.

Now let us recall the definition of homoclinic cycle which is a generalisation of homoclinic orbit. Here we implicitly assume that ω has only finitely many critical points. For a more general treatment see [23], where a more general definition of limit set is considered. For a general definition of limit set, we refer to page 28 in [20].

Definition 6.3.6 A sequence of trajectories $\{\gamma_i(t) : \mathbb{R} \to M\}_{1\leq i\leq n}$ on a manifold M is called a *homoclinic cycle of length* n if for each γ_i its limit $\lim_{t\to\pm\infty}\gamma_i(t)$ exists and satisfies:

$$\lim_{t\to+\infty}\gamma_i(t) = \lim_{t\to-\infty}\gamma_{i+1}(t) \text{ for } 1\leq i\leq n-1, \text{ and } \lim_{t\to+\infty}\gamma_n(t) = \lim_{t\to-\infty}\gamma_1(t).$$

Let ω be a closed 1-form on M, we want to measure the displacement of a trajectory by ω:

Definition 6.3.7 A trajectory γ is said to have *displacement* N *by* ω if its integral with respect to ω equals N:

$$\int_\gamma \omega = N,$$

a homoclinic cycle $\{\gamma_i\}$ has *displacement* N *by* ω if

$$\sum_i \int_{\gamma_i} \omega = N.$$

6.3.1 Main result

Now we are ready to state the theorem:

Theorem 6.3.8 Let M be a smooth compact Riemannian manifold with boundary ∂M, and ω be a closed 1-form on M with exit set $B \subset \partial M$ satisfying assumptions $\mathbf{A1}, \mathbf{A2}$ and $\mathbf{A3}$, if the number of critical points of ω is less than $\mathrm{cat}^0(M, B, [\omega])$, then any gradient-like vector field of ω contains at least one homoclinic cycle.

For the sake of convenience of the proof, we rephrase the theorem as follows:

Theorem 6.3.9 Let M be a smooth compact Riemannian manifold with boundary ∂M, and ω be a closed 1-form on M with exit set $B \subset \partial M$ satisfying assumptions

A1, **A2** and **A3**, suppose there exists a gradient vector field of ω without homoclinic cycles, then:

$$\operatorname{cat}^0(M, B, [\omega]) \leq \text{ number of critical points of } \omega$$

.

Proof: Suppose the gradient vector field of ω we choose satisfies the hypothesis, i.e. for any $N > 0$ there exists no homoclinic cycle with displacement less than N, and the number of critical points of ω is k. For some $C > 0$ and any such $N > 0$ we need to show the existence of an open cover $M = U \cup U_1 \cup \cdots \cup U_k$ with U N movable relative to B and U_i null-homotopic. Here $\xi = [\omega] \in H^1(M; \mathbb{R})$ is the cohomology class of ω.

The idea is to use the negative gradient flow as prototype for the homotopies and partition the manifold according to the destination of each point travelling along its flow line.

Because the homotopy is modified from the negative gradient flow, the integral $\int \omega \leq 0$ is always non-positive along trajectories, so we can choose control $C = 0$. Let us fix $N > 0$, we want to construct an open cover of M as

$$M = U \cup U_1 \cup \cdots \cup U_k.$$

We firstly define U as the open subset of all the points either reach B in finite time or travel over displacement N in the negative direction:

$$U = \{x \in M : \text{ there exists some } t_x > 0 \text{ such that either } x \cdot t_x \in B, \text{ or } \int_x^{x \cdot t_x} \omega < -N\}.$$

Secondly, for U_i, we first need a so-called *gradient-convex neighbourhood* V_i for each critical points p_i, in order to construct open subsets. For each critical point p_i, the gradient-convex neighbourhood V_i is a small closed disc containing p_i, such that the points on the boundary of V_i who are leaving V_i have to travel over displacement N before returning to $\operatorname{Int} V_i$. The existence of V_i is derived from the no homoclinic cycle condition in the hypothesis, for the sake of completeness, we provide a more detailed argument in Section 6.3.2 below, similar to [12] and [23]. Then we define U_i for each p_i as follows:

$$U_i = \{x \in M : x \cdot t_x \in \operatorname{Int} V_i \text{ for some } t_x \in \mathbb{R} \text{ and } \int_x^{x \cdot t_x} \omega > -N\}.$$

The null homotopy of U_i can also be found proved in [12] and [23], which we also supplement in Section 6.3.2.

Now we are left to show the movability of U. The subset U is open since it is the union of two open subsets, namely $\{x \in M : \int_x^{x\cdot t_x} \omega < -N$ for some $t_x > 0$, where $N > 0\}$ and $\{x \in M : x \cdot t_x \in B$ for some $t_x > 0\}$, they are both open according to some implicit function theorem argument and Lemma 6.3.3.

According to Lemma 6.3.4 and the Implicit Function Theorem, for each $x \in U$, there exists $t_x \in \mathbb{R}$, such that either $x \cdot t_x \in B$ or $\int_x^{x\cdot t_x} \omega = -N$. Moreover, the map $\beta : x \to t_x$ is a real continuous function on U. Therefore, we can define the homotopy $h : U \times [0,1] \to M$ as

$$h(x, \tau) = x \cdot (\tau\beta(x)).$$

Together with results in the rest of the section, we have proved the statement.□

To finish this section, we first supplement a proof showing the equality between $\mathrm{cat}^0(X, B, \xi)$ and $\mathrm{cat}(X, B, \xi)$:

Theorem 6.3.10 For any CW pair (X, B),

$$\mathrm{cat}^0(X, B, \xi) = \mathrm{cat}(X, B, \xi)$$

Proof: According to the definitions, since $\mathrm{cat}^0(X, B, \xi)$ requires B to be fixed, which is extra to the definition of $\mathrm{cat}(X, B, \xi)$, so any open cover works for $\mathrm{cat}^0(X, B, \xi)$ would work for $\mathrm{cat}(X, B, \xi)$ as well, therefore, $\mathrm{cat}^0(X, B, \xi) \geq \mathrm{cat}(X, B, \xi)$.

Now to show the inequality in the other direction, the idea is that we extend X to a slightly bigger CW-complex X^+ by attaching an external collar to B, namely $B \times [0, 1]$, likewise, extend the homotopy in definition 6.1.11 to a bigger subset containing $B \times [0, 1]$, and by showing the homotopy equivalence of the pairs $(X, B) \simeq (X^+, B \times \{1\})$, we shall prove $\mathrm{cat}^0(X, B, \xi) \leq \mathrm{cat}(X, B, \xi)$.

We divide the details into two parts: homotopy equivalence of $(X, B) \simeq (X^+, B \times \{1\})$ and then the inequality $\mathrm{cat}^0(X, B, \xi) \leq \mathrm{cat}(X, B, \xi)$:

(i) $(\mathbf{X}, \mathbf{B}) \simeq (\mathbf{X}^+, \mathbf{B} \times \{\mathbf{1}\})$

Let $B \times [0, 1]$ be an external collaring of B by identifying $B \times \{0\}$ with B in X, so we now have a bigger CW complex $X^+ = X \cup_{B\times\{0\}} B \times [0, 1]$, similarly, we denote $U^+ = U \cup_{B\times\{0\}} B \times [0,1]$ and $B^+ = B \times [0,1]$. Then we have a natural deformation retraction $D : X^+ \times [0,1] \to X^+$ of X^+ while keeping $B \times \{1\}$ fixed as the homotopy extension of the homotopy $d : B^+ \times [0,1] \to B^+$ which is

$$d(b,s,t) = (b,(1-t)s+t).$$

Notice $d(b,s,0) = (b,s) = \mathrm{id}_{B^+}(b,s)$ and $d(b,s,1) = (b,1) \in B \times \{1\}$, i.e. d fixes $B \times \{1\}$. Because of the homotopy extension property of (X^+, B^+), D is well-defined and fixes $B \times \{1\}$ as well as d.

Now let $r : B \times [0,1] \to B$ be the projection, we define the map $r^+ : (X^+, B \times \{1\}) \to (X, B)$ as

$$r^+ = \begin{cases} x & x \in X \\ r(x) & x \in B^+, \end{cases}$$

and we show r^+ is a homotopy equivalence by constructing two homotopies, namely

$$F : r^+ j \simeq \mathrm{id}_X \text{ rel } B$$

and

$$H : j r^+ \simeq \mathrm{id}_{X^+} \text{ rel } B \times \{1\},$$

where $j = D_1 i : (X,B) \to (X^+, B\times\{1\})$ composite with inclusion $i : X \to X^+$. Let $F = r^+ D(i \times \mathrm{id}) : X \times [0,1] \to X$, then

$$F(x,0) = r^+ D(i(x),0) = r^+(i(x)) = \mathrm{id}_X(x)$$

and

$$F(x,1) = r^+ D(i(x),1) = r^+(D_1(i(x))) = r^+ j(x).$$

Also $F(b,0,t) = r^+ D(i(b,0,t) = r^+(b,t) = b$ shows B is fixed in the homotopy. Therefore

$$F : r^+ j \simeq \mathrm{id}_X \text{ rel } B.$$

For the existence of $H : jr^+ \simeq \mathrm{id}_{X^+}$ rel $B \times \{1\}$, since $jr^+ = D_1 i r^+$, we get H by concatenating two homotopies $R : ir^+ \simeq \mathrm{id}_{X^+}$ and $h : D_1 \simeq \mathrm{id}_{X^+}$. Easily, we obtain $h : X^+ \times [0,1] \to X^+$ by reversing the homotopy D:

$$h(x,t) = D(x, 1-t);$$

and $R : X^+ \times [0,1] \to X^+$ is defined as

$$R(x,t) = \begin{cases} x & x \in X \\ (b, st) & x = (b,s) \in B^+ = B \times [0,1], \end{cases}$$

so $R(b,s,0) = (b,0) = b = ir(b,s)$ and $R(b,s,1) = (b,s) = \mathrm{id}_{X^+}$, and $R(b,0) = (b,0) \in X \cap B \times [0,1]$ shows the continuity of R. However R does not fix $B \times \{1\}$, but by composite with D_1, $D_1 \circ R : X^+ \times [0,1] \to X^+$ recovers the condition as

$$D_1 R(b,1,t) = D_1(b,t) = d(b,t,1) = (b,1).$$

So we have $H : X^+ \times [0,1] \to X^+$ rel $B \times \{1\}$ as

$$H(x,t) = \begin{cases} D_1 R(x, 2t) & 0 \le t \le \frac{1}{2} \\ h(x, 2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

It is true that $H(x,0) = D_1 R(x,0) = D_1 i r^+ = (D_1 i) r^+ = j r^+$ and $H(x,1) = h(x,1) = D(x,0) = \mathrm{id}_{X^+}$. Therefore

$$H : j r^+ \simeq \mathrm{id}_{X^+} \text{ rel } B \times \{1\}.$$

(ii) $\mathrm{cat}^{\mathbf{0}}(\mathbf{X}, \mathbf{B}, \xi) \le \mathrm{cat}(\mathbf{X}, \mathbf{B}, \xi)$

Since we are talking about finite CW complex, the compactness of X ensures the existence of some real number $K \in \mathbb{R}$ such that

$$\int_x^{D_1(x)} \omega \le K \text{ for all } x \in X,$$

where ω lies in the cohomology class $\xi \in H^1(X, \mathbb{R})$. Suppose $\mathrm{cat}(X, B, \xi) = k$ and fix $N > 0$, then for $N + K$, we have an open cover of X as $X = U \cup U_1 \cup \cdots \cup U_k$ and let $\xi' = r^{+*}(\xi) \in H^1(X^+, \mathbb{R})$ and choose a continuous closed 1-form ω' on X^+ representing ξ', $[\omega'] = \xi'$. We want to show the homotopy $g : U \times [0,1] \to X \overset{i}{\hookrightarrow} X^+$ satisfying condition (b) for $\mathrm{cat}(X, B, \xi)$ can be extended to some $G : U^+ \times [0,1] \to X^+$ rel $B \times \{1\}$, such that for any $x \in U^+$ either

$$G_1(x) \in B \times \{1\}$$

or

$$\int_x^{G_1(x)} \omega' < -N,$$

and for all $t \in [0,1]$,

$$\int_x^{G_1(x)} \omega' < C' \text{ for some } C'.$$

Now consider the restriction $g_t|_B : B \to X$ of g_t in B with $g_t(b) \in B$ for all t. Firstly, we want to extend this map to B^+ as $g' : B^+ \times [0,1] \to X^+$ such that $g'|_{B\times\{0\}} = g$ for the continuity and $g'|_{B\times\{1\}} = \mathrm{id}$. For any $(b,s,t) \in B^+ \times [0,1]$,

$$g'(b,s,t) = (g(b,(1-s)t), s)$$

delivers the satisfaction, as $g'(b,0,t) = (g(b,t),0) = g(b,t) \in B$ and $g'(b,1,t) = (g(b,0),1) = (b,1) = \mathrm{id}(b,1)$.

Next, define $g^+ : U^+ \times [0,1] \to X^+$ as

$$g^+(x,t) = \begin{cases} g(x,t) & x \in U \\ g'(x,t) & x \in B^+. \end{cases}$$

However, since $g'_t(b,s) = (*,s)$ keeps any point $(b,s) \in B \times [0,1]$ in the same level as t varies, i.e. g^+ has not been exactly the homotopy driving points into $B \times \{1\}$ or N distance, we need to modify it one more time:

Define $G : U^+ \times [0,1] \to X^+$ as

$$G(x,t) = \begin{cases} g^+(x,2t) & 0 \le t \le \frac{1}{2} \\ D(g^+1(x), 2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Now let $C' = C + K$, G satisfies condition (b) of the Definition 6.3.1, therefore,

$$\mathrm{cat}^0(X,B,\xi) \le \mathrm{cat}(X,B,\xi).$$

We have proved the equality. $\square$

6.3.2 Proof of null homotopy of U_i in Theorem 6.3.9

Lemma 6.3.11 Let ω be a closed 1-form on a manifold M, for any neighbourhood U of a critical point p of ω, there exists a closed neighbourhood $V \subset U$ of p , such that:

1. For any $x \in M$, $I_x = \{t : x \cdot t \in V\}$ is convex, i.e. I_x is either empty or a closed interval $[a_x, b_x]$ possibly half infinite or degenerate to a point: $a_x = b_x$.
2. If I_x is nonempty, then the function $x \to a_x$ is continuous.

Proof: This is a local problem so we can simplify it to a closed 1-form with only one critical point p. Suppose further that there exists $f : U \to \mathbb{R}$ such that $\omega|_U = \mathrm{d}f$ and $f(p) = 0$. Let $\epsilon > 0$ be small such that the neighbourhood $N_\epsilon = N_\epsilon(p) = \{x : \mathrm{dist}(x, p) \leq \epsilon\} \subset U$, then the condition (1) is equivalent to the existence of $\delta > 0$ such that the trajectory γ_x of any $x \in U$ with minimal distance $\mathrm{dist}(\gamma_x \cap f^{-1}([-\delta, \delta]), p) \leq \frac{\epsilon}{2}$, we have

$$\gamma_x \cap f^{-1}([-\delta, \delta]) \subset N_\epsilon.$$

For such δ enables the construction of V to be the closure of

$$O_\epsilon = \{x \in f^{-1}((-\delta, \delta)) : \mathrm{dist}(\gamma_x \cap f^{-1}([-\delta, \delta]), p) \leq \frac{\epsilon}{2}\}.$$

Notice $V \subset N_\epsilon$ and for any $x \in M$ with $x \cdot a_x \in V$ for some $a_x \in \mathbb{R}$, then γ_x will be in N_ϵ until some $b_x > a_x \in \mathbb{R}$, where $x \cdot t \notin V$ and $f(x \cdot t) < -\delta$ if $t > b_x$. And $[a_x, b_x]$ is convex.

Now we want to show such δ exists. Suppose there is no such δ, then we have a collection of points $\{x_k\} \in N_\epsilon$ with $f(x_k) \in [-\frac{1}{k}, \frac{1}{k}]$ and $\mathrm{dist}(x_k, p) \leq \frac{\epsilon}{2}$ such that there exists t_k for each x_k with $x_k \cdot [0, t_k] \subset f^{-1}[-\frac{1}{k}, \frac{1}{k}]$ and $\mathrm{dist}(x_k \cdot t_k, p) > \epsilon$. Since N_ϵ is compact, we can choose a subsequence of $\{x_k\}$ converges to some point $x \in N_\epsilon$. On the other hand, since $\mathrm{dist}(x_k, x_k \cdot t_k) \geq \frac{\epsilon}{2}$ and the gradient vector field v of ω is nontrivial away from p in $N_\epsilon \setminus N_{\frac{\epsilon}{2}}$, t_k are then strictly positive. So choose such $t > 0 \in \{t_k\}$, notice that

$$x_k \cdot [0, t] \to x \cdot [0, t] \in \bigcap_{k \geq 1} f^{-1}[-\frac{1}{k}, \frac{1}{k}] = f^{-1}(0),$$

which contradicts to the fact that

$$(f \cdot \gamma)'(t) = \mathrm{d}f(\gamma(t))\gamma'(t) = -\mathrm{d}f(\gamma(t))(v(\gamma(t))) = -\|v\|^2 < 0 \tag{6.2}$$

For the second property, we firstly denote

$$\partial_- V = V \cap f^{-1}(-\delta) = \{x \in \partial V : \text{ for some small } t > 0,\ x \cdot t \notin V\}$$

and

$$\partial_+ V = V \cap f^{-1}(\delta) = \{x \in \partial V : \text{ for some small } t < 0,\ x \cdot t \notin V\},$$

then for each x with $\gamma_x \cap V \neq \emptyset$, using (1) above and the Implicit Function Theorem, there exists $a_x \in \mathbb{R}$ such that $x \cdot a_x \in \partial_+ V$ and open neighbourhood $U_x \ni x$ and $I_{a_x} \ni a_x$ such that there exists continuous function $\psi : U_x \to I_{a_x}$ with $\psi(x) = a_x$. So we proved the continuity. □

Lemma 6.3.12 Let ω be the closed 1-form on a compact manifold M, and v be a gradient vector field of ω, which produces no homoclinic cycle in M with displacement by ω less than $N > 0$, and for any $x \in \partial_- V = V \cap f^{-1}(-\delta) = \{x \in \partial V :$ for some small $t > 0, x \cdot t \notin V\}$ defined as above, such that there exists some $t > 0$ with $x \cdot t \in \operatorname{Int} V$, then

$$\int_x^{x \cdot t} \omega < -N.$$

What Lemma 6.3.12 says is that when a point exits the convex neighbourhood V through $\partial_- V$, it has to travel longer than N distance before returning to V.

Proof: Suppose the contrary, then there exists a sequence of points $x_{i,n} \in \partial_- V_i$ and sequences of real numbers $t_{i,n} > 0$ and $s_{i,n} < 0$ such that:

(a) $x_{i,n} \cdot [s_{i,n}, 0] \subset V_i$ and $\operatorname{dist}(x_{i,n} \cdot s_{i,n}, p_i) < \frac{1}{n}$;

(b) $\operatorname{dist}(x_{i,n} \cdot t_{i,n}, p_i) < \frac{1}{n}$;

(c) $\int_{x_{i,n}}^{x_{i,n} \cdot t_{i,n}} \omega \geq -N$.

Pass $\{x_{i,n}\}$ to a subsequence, it converges to some point x_i, so are $\{t_{i,n}\}$ and $\{s_{i,n}\}$ to some $\{t_i\}$ and $\{s_i\}$, respectively. According to the invariance of p_i, as n gets larger, it takes longer time to get closer to p_i, (a) and (b) together with the continuity of the flow, we conclude that $t_i = \infty$ and $s_i = -\infty$, moreover, (a) tells us that $\mathrm{dist}(x_i \cdot s_{i,n}, p_i) \to 0$ as $n \to \infty$, i.e. $x_i \cdot s_{i,n} \to p_i$ as $s_{i,n} \to s_i = -\infty$.

Now we want to understand what happens to $x_i \cdot t_{i,n}$ as $t_{i,n} \to t_i = \infty$. We claim it approaches some critical point p_j. Suppose the contrary, then there exists $\epsilon > 0$ such that $N_\epsilon(p_j) = \{x \in M : \mathrm{dist}(x, p_j) \leq \epsilon\} \cap x_i \cdot t_{i,n} = \emptyset$, for all $j = 1, \ldots, k$ and all $t_{i,n}$. Then there exists $c > 0$ such that

$$\|v\|^2_{X - \bigcup_j N_\epsilon(p_j)} \leq -c,$$

i.e. $x_i \cdot t_{i,n}$ will keep flowing further and further in the negative direction, unboundedly. This is contradictory to (c): $\int_{x_{i,n}}^{x_{i,n} \cdot t_{i,n}} \omega \geq -N$. Therefore, $x_i \cdot t_{i,n} \to p_j$ as $t_{i,n} \to t_i = \infty$.

Now if $j = i$, then we are finished with a homoclinic cycle with displacement shorter that N, contradicting to the hypothesis in the lemma. If $j \neq i$, let V_j be the convex neighbourhood of p_j, let T_j be the entry time of x_i, i.e. $x_i \cdot T_j \in \partial_+ V_j$. Then for n large, $\mathrm{dist}(x_{i,n} \cdot T_j, \partial_+ V_j) \to 0$, the points $x_{i,n}$ spend similar time reaching $\partial_+ V_j$, for such $x_{i,n}$, as its destination is p_i, it has to exit v_j by some finite time $t'_{i,n}$, i.e. there exists some point $x'_{i,n} = x_{i,n} \cdot t'_{i,n} \in \partial_- V_j$. Pass such $\{x'_{i,n}$ to a subsequence converging to some x'_o. Notice $t'_{i,n} - T_j \to \infty$ as $n \to \infty$. Therefore similar argument leads us to $x'_i \cdot t \to p_j$ as $t \to -\infty$ and $x'_i \cdot t \to p_l$ as $t \to \infty$ for another critical point p_l.

Repeating this same argument for p_l and so on, after at most k times, there will be some critical point appears twice for the limits of the sequences of trajectories, which forms a homoclinic cycle. We run into contradiction again. □

Such disk V_p we constructed for a critical point p in the lemmas is called *gradient-convex neighbourhood* of p. We can rephrase the definition of U_i for each critical point p_i of closed 1-form ω with this notion:

$$U_i = \{x \in M : x \cdot t_x \in \mathrm{Int}\, V_i \text{ for some } t_x \in \mathbb{R} \text{ and } \int_x^{x \cdot t_x} \omega > -N\}.$$

A null homotopy for U_i is equivalent to a homotopy mapping U_i into V_i, and we shall finish the proof by constructing such homotopy h_i for each subset U_i. Firstly, one can define a continuous function $\phi_i : U_i \to \mathbb{R}$ from U_i to the real as follows:

$$\phi_i(x) = \begin{cases} 0 & x \in \operatorname{Int} V_i \\ a_x & x \in U_i - \operatorname{Int} V_i. \end{cases}$$

To show ϕ_i is continuous is equivalent to show it is sequentially continuous. Suppose we have a sequence of points $\{x_n\}$ in U_i and it converges to some x^*, we want to show $\phi_i(x_n) \to \phi_i(x^*)$ as $n \to \infty$.

In the case that $x^* \in V_i$, it is not possible that $x^* \in \partial_- V_i$ as it would travel longer than N distance before returning to $\operatorname{Int} V_i$, violating the definition of U_i. So there exists some $\epsilon > 0$ such that $x^* \cdot (0, \epsilon) \subset \operatorname{Int} V_i$, in particular $x^* \cdot \frac{1}{n} \in \operatorname{Int} V_i$ for any $n \in \mathbb{R}$. Now observe that for large n and very small ϵ, $x_n \cdot (0, \epsilon) \subset \operatorname{Int} V_i$, for otherwise, there exists $\epsilon'_n > 0$ for each n, such that $x_n \cdot (0, \epsilon'_n) \cap V_i = \emptyset$, i.e. $\operatorname{dist}(x_n \cdot \epsilon'_n, x^*) \geq \operatorname{dist}(x_n \cdot \epsilon'_n, V_i)$. So we can choose $\delta < \min\{\operatorname{dist}(x_n \cdot \epsilon'_n, V_i)\}$, then the δ-ball $B_\delta(x^*)$ contains no x_n and x^* is not a limit point. Contradiction. Therefore, $a_{x_n} \in (0, \epsilon)$ and particularly $a_{x_n} < \frac{1}{n}$ for each n, so $a_{x_n} \to 0$ as $n \to \infty$.

For the second case that $x^* \notin V_i$, the second property of Lemma A.1 provides the continuity for the map $x^* \to a^*$ which in turn shows the sequential continuity in this context: $x_n \to x^*$ then $a_n \to a^*$.

At the end, we define the homotopy $h_i : U_i \times [0, 1] \to M$ readily:

$$h_i(x, t) = x \cdot (\phi(x)t)$$

.

This completes the proof. □

Chapter 7

Conclusion

The main motivation of this research project is to present a geometric report of the relative homological information a closed 1-form reveals on manifolds with boundary. We were aiming to achieve parallel conclusions stated in the literature, e.g. [4] which takes on an analytic approach, but failed in the case of more degenerate settings, namely, in the sense of Kirwan. For the stratification techniques in [20] are not directly applicable in our situation.

Overall, we managed to generalise the results to Bott nondegeneracy with the no homoclinic cycle condition imposed, and tackled more degenerate situations via the Lusternik-Schnirelman category. The no homoclinic cycle condition seems essential for our construction but is absent in other approaches, such as [13, Chapter 5]. In the future, we wish to have better understanding of this obstruction and its links to the category with respect to a closed 1-form generalised in [23].

Bibliography

[1] M. Akaho, *Morse homology and manifolds with boundary*, Commun. Contemp. Math. **9**(2007), no. 3, 301–334.

[2] D. Austin and P. Braam, *Morse-Bott Theory and Equivariant Cohomology*, The Floer memorial volume, Progr. Math., **133**(1995), Birkhuser, Basel, 123–183.

[3] R. Abraham and J. Robbin, *Transversal Mappings and Flows*, W. A. Benjamin, Inc., New York-Amsterdam 1967

[4] M. Braverman and V. Silantyev, *Kirwan-Novikov Inequalities on a Manifold with Boundary*, Trans. Amer. Math. Soc. **358**(2006), no. 8, 3329–3361.

[5] G. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics, **139**, Springer-Verlag, New York, 1997.

[6] O. Cornea, *Some properties of the relative Lusternik-Schnirelmann category*, Stable and unstable homotopy(Toronto, ON, 1996), 67–72.

[7] O. Cornea, G. Lupton, J. Oprea and D. Tanré, *Lusternik-Schnirelmann category*, AMS, 2003.

[8] O. Cornea and A. Ranicki, *Rigidity and Gluing for Morse and Novikov Complexes*, J. Eur. Math. Soc. (JEMS) **5**(2003), no. 4, 343–394.

[9] M. Farber, *Dirichlet units and critical points of closed 1-forms*, C. R. Acad. Sci. Paris, **328**(1999), 695–700.

[10] M. Farber, *Lusternik-Schnirelman theory for closed 1-forms*, Commentarii Math, Helverici, **75**(2000), 156–170.

[11] M. Farber *Topology of closed 1-forms and their critical points*, Topology, **40**(2001), 235–258.

[12] M. Farber, *Zeros of closed 1-forms, homoclinic orbits, and Lusternik-Schnirelman theory*, Topological Methods in Nonlinear Analysis, **19**(2002), 123–152.

[13] M. Farber, *Topology of Closed One-Forms*, Mathematical Surveys and Monographs, **108**, American Mathematical Society, Providence, RI, 2004.

[14] M. Farber and D. Schütz, *Cohomological estimates for* $\mathrm{cat}(X, \xi)$, Geom. Topol. **11**(2007), 1255–1288.

[15] M. Farber and D.Schütz, *Moving Homology Classes to Infinity*, Forum Math. **19**(2007), no. 2, 281–296

[16] K. Fukaya, *Floer Homology of Connected Sum of Homology 3-Spheres*, Topology, **35** (1996), no.1, 89–136.

[17] A. Hatcher, *Algebraic Toplogy*, Cambridge University Press, Cambridge, 2002.

[18] M. Hatchings, *Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves).*

[19] I. M. James, *On the category in the sense of Lusternik-Schnirelman*, Topology, **17**(1978), 331–348.

[20] F. Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, Mathematical Notes, **31**, Princeton University Press, Princeton, NJ, 1984.

[21] I. Kupka, *Contribution la thorie des champs gnriques.* Contributions to Differential Equations **2**(1963) 457–484.

[22] F. Latour, *Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham*, Inst. Hautes tudes Sci. Publ. Math. **80**(1994), 135–194.

[23] J.Latschev: *Flows with Lyapunov one-forms and a generalization of Farber's theorem on homoclinic cycles*, Int. Math. Res. Not. **5**(2004), 239–247.

[24] John M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics, **218**, Springer-Verlag, New York, 2003.

[25] L. Lusternik, *Topologische Grundlagen der allgemeinen Eigenwerttheorie*, Monatsh. Math. Phys. **37**(1930), no. 1, 125–130.

[26] L. Lusternik and L. Schnirelman, *Topological methods in variational problems and their application to the differential geometry of surfaces*, (Russian) Uspehi Matem. Nauk (N.S.) **2**(1947). no. 1(17), 166–217.

[27] J. McCleary, *User's Guide to Spectral Sequences*, Cambridge Studies in Advanced Mathematics, **58**, Cambridge University Press, Cambridge, 2001.

[28] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, **51**, 12–24.

[29] J. Milnor, *Lectures on the h-Cobordism Theorem*, Princeton University Press, Princeton, N.J. 1965.

[30] P. M. Moyaux, *Lower bounds for the relative Lusternik-Schnirelmann category*, Manuscripta Math. **101**(2000), no. 4, 533–542.

[31] J. Munkres, *Elementary Differential Topology*, Annals of Mathematics Studies, **54**, Princeton University Press, Princeton, N.J. 1966.

[32] S.P. Novikov, *Multi-valued functions and functionals. An analogue of Morse theory*, Soviet Math. Doklady, **24**(1981), 222–226.

[33] S.P. Novikov, *The Hamiltonian formalism and a multi-valued analogue of Morse Theory*, Russian Math. Surveys, **37:5**(1982), 1–56.

[34] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Mordern Geometry- Methods and Applications: Part II The Geometry and Topology of Manifolds.*

[35] A.v. Pajitnov, *Circle-valued Morse Theory*, de Gruyter Studies in Mathematics, **32**, Walter de Gruyter Co., Berlin, 2006.

[36] M. Poźniak, *Floer homology, Novikov rings and clean intersections*, Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2, **196**(1999), Amer. Math. Soc. 119–181.

[37] M. Reeken, *Stability of Critical Points under Small Perturbations, Part I: Topological Theory*, Manuscripta Math. **7**(1972), 387–411.

[38] D. Schütz, *Geometric chain homotopy equivalences between Novikov complexes*, High-dimensional manifold topology, 469–498, World Sci. Publ., River Edge, NJ, 2003

[39] S. Smale, *Stable manifolds for differential equations and diffeomorphisms.* Ann. Scuola Norm. Sup. Pisa (3), **17**(1963), 97–116.

[40] S. Smale, *On gradient dynamical systems.* Ann. of Math. (2) **74**(1961) 199–206.

[41] E. Spanier, *Algebraic Topology*

[42] C. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, **38**, Cambridge University Press, Cambridge, 1994.

[43] M.E. Zadeh, *Morse inequalities for manifolds with boundary*(preprint)